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Abstract We detail and theoretically analyse the so-called *fast vector (or velocity) penalty-projection methods (VPP$_\varepsilon$)* of which the main ideas and features are briefly introduced in [8,9,10]. This family of numerical schemes proves to efficiently compute the solution of unsteady Navier-Stokes/Brinkman problems governing incompressible or low Mach multiphase viscous flows with variable mass density and/or viscosity or anisotropic permeability. In this paper, we describe in detail the connections and essential differences with usual methods to solve the Navier-Stokes equations.

The key idea of the basic (VPP$_\varepsilon$) method is to compute at each time step an accurate and curl-free approximation of the pressure gradient increment in time. This is obtained by proposing new Helmholtz-Hodge decomposition solutions of L^2-vector fields in bounded domains to get fast methods with suitable adapted right-hand sides; see [11]. This procedure only requires a few iterations of preconditioned conjugate gradients whatever the spatial mesh step. Then, the splitting (VPP$_\varepsilon$) method performs a *two-step approximate divergence-free vector projection* yielding a velocity divergence vanishing as $\mathscr{O}(\varepsilon\,\delta t)$, δt being the time step, with a penalty parameter ε as small as desired until the machine precision, e.g. $\varepsilon = 10^{-14}$, whereas the solution algorithm can be extremely fast and cheap. Indeed, the proposed *velocity correction step* typically requires only one or two iterations of a suitable preconditioned Krylov solver whatever the spatial mesh step [10]. Moreover, the robustness of our method is not sensitive to large mass density ratios since the velocity penalty-projection step does not include any spatial derivative of the density.

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In the present work, we also prove the theoretical foundations as well as global solvability and optimal unconditional stability results of the (VPP$_\varepsilon$) method for Navier-Stokes problems in the case of homogeneous flows, which are the main new results.

Keywords Vector penalty-projection method · divergence-free penalty-projection · penalty method · splitting prediction-correction scheme · fast Helmholtz-Hodge decompositions · Navier-Stokes/Brinkman equations · stability analysis · incompressible homogeneous flows · dilatable flows · low Mach number flows · incompressible non-homogeneous or multiphase flows

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1 Introduction to the mathematical models

Notations. We use below the usual functionnal setting for the unsteady Navier-Stokes equations, see [53,54,51,55,75,40,20]. Let $\Omega \subset \mathbb{R}^d$ ($d=2$ or 3 in practice) be an open bounded and connected set with a Lipschitz continuous boundary $\Gamma = \partial\Omega$ and $\mathbf{n}$ be the outward unit normal vector on Γ. Due to some further technicalities, we also assume that either Γ is of class $\mathscr{C}^{1,1}$ or Ω is a convex domain.

In particular, we use $\|.\|_0$ for the $L^2(\Omega)$-norm, $\|.\|_1$ for the $H^1(\Omega)$-norm, $\|.\|_{-1}$ for the $H^{-1}(\Omega)$-norm, $(.,.)_0$ for the $L^2(\Omega)$ inner product and $\langle .,. \rangle_{-1}$ for the duality pairing between $H^{-1}(\Omega)$ and $H^1_0(\Omega)$.

We define below some Hilbert spaces with their usual respective inner products and associated norms:

$$
\begin{aligned}
H_{div}(\Omega) &= \left\{ \mathbf{u} \in L^2(\Omega)^d;\ \nabla\cdot\mathbf{u} \in L^2(\Omega) \right\} \\
H_{div,0}(\Omega) &= \left\{ \mathbf{u} \in L^2(\Omega)^d;\ \nabla\cdot\mathbf{u} \in L^2(\Omega),\ \mathbf{u}\cdot\mathbf{n} = 0 \text{ on } \Gamma \right\} \\
H &= \left\{ \mathbf{u} \in L^2(\Omega)^d;\ \nabla\cdot\mathbf{u} = 0,\ \mathbf{u}\cdot\mathbf{n} = 0 \text{ on } \Gamma \right\} \\
H_{rot}(\Omega) &= \left\{ \mathbf{u} \in L^2(\Omega)^d;\ \nabla\times\mathbf{u} \in L^2(\Omega)^d \right\} \\
H^1_n(\Omega) &= \left\{ \mathbf{u} \in H^1(\Omega)^d;\ \mathbf{u}\cdot\mathbf{n} = 0 \text{ on } \Gamma \right\} \\
L^2_0(\Omega) &= \left\{ q \in L^2(\Omega);\ \int_\Omega q\,dx = 0 \right\}.
\end{aligned}
$$

For $T > 0$, we consider the following unsteady Navier-Stokes/Brinkman problem [50, 23] governing incompressible non-homogeneous or multiphase flows with Dirichlet boundary conditions for the velocity $\mathbf{v}_{|\Gamma} = 0$ on Γ. The force term $\mathbf{f} \in L^2(0,T;H^{-1}(\Omega)^d)$ and initial data $\mathbf{v}(t=0) = \mathbf{v}_0 \in H$, $\varphi(t=0) = \varphi_0 \in L^\infty(\Omega)$ with $\varphi_0 \geq 0$ *a.e.* in Ω, are given. We focus on the model problem (1-3), as a part of more complex fluid mechanics problems, e.g. [52,56], written below for isothermal configurations:

$$\rho\left(\partial_t \mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v}\right) - 2\nabla\cdot(\mu\,\mathbf{d}(\mathbf{v})) + \mu\,\mathbf{K}^{-1}\,\mathbf{v} + \nabla p = \mathbf{f} \quad \text{in } \Omega\times(0,T) \tag{1}$$

$$\nabla\cdot\mathbf{v} = 0 \quad \text{in } \Omega\times(0,T) \tag{2}$$

$$\partial_t\,\varphi + \mathbf{v}\cdot\nabla\varphi = 0 \quad \text{in } \Omega\times(0,T) \tag{3}$$

where $\mathbf{d}(\mathbf{v})$ denotes the strain rate tensor:

$$\mathbf{d}(\mathbf{v}) = \frac{1}{2}\left(\nabla\mathbf{v} + (\nabla\mathbf{v})^T\right).$$

The permeability tensor $\mathbf{K}$ in the Darcy's drag term is supposed to be symmetric, uniformly positive definite and bounded in Ω. We refer to [50,19,49,23,24] and the references therein for the mathematical or numerical modelling of flows inside complex fluid-porous-solid heterogeneous systems with the Navier-Stokes/Brinkman or Darcy equations. Let us mention [22,4,6,65,50,5] for the mathematical analysis and numerical validations of the fictitious domain model using the so-called L^2 or H^1 volume penalty methods to take account of porous or solid obstacles in flow problems with the Navier-Stokes/Brinkman equations.

The equation (3) for the positive phase function φ governs the transport by the flow of the interface Σ, either between two fluid phases, or between fluid and solid phases, respectively in the case of two-phase fluid flows or fluid-structure interaction problems. A level-set function can be used as well. The previous set of equations must be supplemented by some given state laws of the form:

$$\rho = \rho(\varphi) \quad \text{and} \quad \mu = \mu(\varphi)$$

which are given for both the density and viscosity fields. For example, we use in the numerical results for non-miscible two-phase flows the following laws with a volume (VOF) or discrete phase function $\varphi \in [0,1]$, the iso-surface $\varphi = 0.5$ denoting the sharp interface Σ separating the two phases:

$$\rho(\varphi) = \rho_1\,(1 - H(\varphi - 0.5)) + \rho_2\, H(\varphi - 0.5)$$
$$\mu(\varphi) = \mu_1\,(1 - H(\varphi - 0.5)) + \mu_2\, H(\varphi - 0.5)$$

where $H(X)$ denotes the value of the Heaviside function equal to 0 for $X < 0$ and 1 for $X \geq 1$. The function φ can be practically a volume of fluid (VOF) function, e.g. VOF-PLIC [80] or SVOF [61], or also a level-set function [58]; see [66,67,77] for some improvements and comparisons.

The force $\mathbf{f}$ may include some volumic forces like the gravity force $\rho\,\mathbf{g}$ as well as the surface tension force $\mathbf{f}_{st}$ to describe the capillarity effects at the phase interfaces. Thus we have:

$$\mathbf{f} = \rho\,\mathbf{g} + \mathbf{f}_{st} = \rho\,\mathbf{g} + \sigma\,\kappa\,\mathbf{n}_{|\Sigma}\,\delta_\Sigma$$

where σ is the surface tension coefficient, κ the local curvature of the interface, $\mathbf{n}_{|\Sigma}$ the outward unit normal to the interface (associated with one of the fluids) and δ_Σ the Dirac measure supported by the interface Σ. Hence, our approach is essentially Eulerian using a non boundary/interface-fitted background mesh with a Lagrangian front-tracking of the sharp interfaces accurately reconstructed on the fixed Eulerian cartesian mesh, see e.g. [50, 78,77,66,67,61,79,19,49] and the references therein.

When non-isothermal configurations are considered, the advection-diffusion equation for the temperature θ must be added:

$$(\rho c_p)\,(\partial_t\,\theta + \mathbf{v}\cdot\nabla\theta) - \nabla\cdot(\lambda\,\nabla\theta) = S_\theta \quad \text{in } \Omega \times (0,T) \tag{4}$$

supplemented by adequate initial and boundary conditions for the temperature or the heat flux which will be precised in the numerical results. In those cases, we also assume some given state laws: $\rho = \rho(\varphi,\theta)$ and $\mu = \mu(\varphi,\theta)$ for each phase, where the functions are continuous and positive, e.g. the state law of an ideal gas. Besides, we assume for sake of simplicity that: $\mu = \mu(\varphi, \theta) \geq \mu_0 > 0$.

The case of nonhomegeneous velocity Dirichlet boundary conditions, for example with $\mathbf{v}_{|\Gamma} = \mathbf{v}_D \in L^2(0,T;H^{\frac{1}{2}}(\Gamma)^d)$ on Γ, also requires some given boundary conditions for φ on the inflow part Γ^- of Γ where $\mathbf{v}_D \cdot \mathbf{n}_{|\Gamma} < 0$. For homogeneous flows with constant density, the initial velocity $\mathbf{v}_0$ can be taken in $L^2(\Omega)^d$. We refer to [71,56,20] for the study of global existence of weak solutions to density-dependent Navier-Stokes problems.

The article is organized as follows. In the next Section 2, we describe in detail the (VPP$_\varepsilon$) method for the solution of non-homogeneous viscous flow problems (1-3), and we discuss the main differences and/or connections with other usual methods. We also justify that the method can be very fast if the penalty parameter ε is chosen sufficiently small by the asymptotic expansion of the discrete solution to the velocity penalty-projection step. The Section 3 is devoted to the basic construction of the method where we state the theoretical foundations and proves the theorems 1 and 2 justifying the name *"vector (or velocity) penalty-projection"* of the method. The last Section 4 before the conclusion deals with the theoretical analysis of the (VPP$_\varepsilon$) method for incompressible and unsteady homogeneous flow problems governed by the Navier-Stokes equations. More precisely, we prove in Theorem 3 the global solvability of the method as well as optimal and unconditional stability results in the case of Navier-Stokes problems with constant density.

The (VPP$_\varepsilon$) methods, analysed in the sequel to solve Darcy or Navier-Stokes problems, are briefly presented in the recent Conference paper [8] and in the short Letter [10].

2 The fast vector-penalty projection method (VPP$_\varepsilon$)

2.1 Description of the method

We present hereafter the two-step vector (or velocity) penalty-projection (VPP$_\varepsilon$) method with a penalty parameter $0 < \varepsilon \ll 1$. For $\varphi^0 = \varphi_0 \in L^\infty(\Omega)$ with $\varphi_0 \geq 0$ *a.e.* in Ω, $\mathbf{v}^0 = \mathbf{v}_0 \in L^2(\Omega)^d$ and $p^0 \in L^2_0(\Omega)$ given, the method reads as below with usual notations for the semi-discrete setting in time, $\delta t > 0$ being the time step.

For all $n \in \mathbb{N}$ such that $(n+1)\,\delta t \leq T$, find $\widetilde{\mathbf{v}}^{n+1}$, $\widehat{\mathbf{v}}^{n+1}$ and $\mathbf{v}^{n+1}$, $p^{n+1} \in L^2_0(\Omega)$, $\varphi^{n+1} \in L^\infty(\Omega)$, such that:

$$\rho^n \left(\frac{\widetilde{\mathbf{v}}^{n+1} - \mathbf{v}^n}{\delta t} + (\mathbf{v}^n \cdot \nabla)\widetilde{\mathbf{v}}^{n+1} \right) - 2\,\nabla \cdot \left(\mu^n \, \mathbf{d}(\widetilde{\mathbf{v}}^{n+1}) \right) + \mu^n \, \mathbf{K}^{-1} \, \widetilde{\mathbf{v}}^{n+1} + \nabla p^n = \mathbf{f}^n \qquad \text{in } \Omega \quad (5)$$

$$\text{with} \quad \widetilde{\mathbf{v}}^{n+1} = 0 \qquad \text{on } \Gamma$$

$$\varepsilon \, \frac{\rho^n}{\delta t} \, \widehat{\mathbf{v}}^{n+1} - \nabla \left(\nabla \cdot \widehat{\mathbf{v}}^{n+1} \right) = \nabla \left(\nabla \cdot \widetilde{\mathbf{v}}^{n+1} \right) \qquad \text{in } \Omega \quad (6)$$

$$\text{with} \quad \widehat{\mathbf{v}}^{n+1} \cdot \mathbf{n} = 0 \qquad \text{on } \Gamma$$

$$\mathbf{v}^{n+1} = \widetilde{\mathbf{v}}^{n+1} + \widehat{\mathbf{v}}^{n+1}, \quad \text{and} \quad \nabla (p^{n+1} - p^n) = -\frac{\rho^n}{\delta t} \, \widehat{\mathbf{v}}^{n+1} \qquad \text{in } \Omega \quad (7)$$

$$\frac{\varphi^{n+1} - \varphi^n}{\delta t} + \mathbf{v}^{n+1} \cdot \nabla \varphi^n = 0 \qquad \text{in } \Omega. \quad (8)$$

For non homogeneous Dirichlet conditions, we have: $\widetilde{\mathbf{v}}^{n+1}_{|\Gamma} = \mathbf{v}^{n+1}_D$ and $\widehat{\mathbf{v}}^{n+1} \cdot \mathbf{n}_{|\Gamma} = 0$. Here $\mathbf{v}^n, p^n$ are desired to be first-order approximations of the exact velocity and pressure solutions $\mathbf{v}(t_n), p(t_n)$ at time $t_n = n\,\delta t$. Since the end-of-step velocity divergence is not exactly zero, the additional spherical part $\lambda \nabla \cdot \mathbf{v} \mathbf{I}$ with $\langle = -2\mu/3$ of the Newtonian stress tensor is included within the dynamical pressure gradient ∇p. Once the equations (5-8) have been solved, the advection-diffusion equation of temperature can be solved too for θ^{n+1}, e.g. with a standard linearly implicit Euler scheme, and we can find: $\rho^{n+1} = \rho(\varphi^{n+1}, \theta^{n+1})$ and $\mu^{n+1} = \mu(\varphi^{n+1}, \theta^{n+1})$.

For the sake of simplicity in the numerical procedure, a semi-implicit scheme where the nonlinear term is linearized by the first term is here chosen since it does not suffer from stability conditions. It can be also treated fully explicitly with a CFL-like stability condition. In any case, a CFL condition must be verified due to the explicit scheme (8) generally used to solve the advection equation for the phase function φ. The error analysis of the fully implicit scheme is generally simpler but it also requires to use a quasi-Newton algorithm to solve the corresponding non linear system at each time step in the practical computations.

Remark 1 (Vector correction for Navier-Stokes/Brinkman problems.)
In order to get a method with a better consistency, specially in the case of variable permeability, the velocity correction (6) and pressure gradient correction (7) steps are respectively replaced by:

$$\varepsilon \left(\frac{\rho^n}{\delta t} + \mu^n \mathbf{K}^{-1} \right) \widehat{\mathbf{v}}^{n+1} - \nabla \left(\nabla \cdot \widehat{\mathbf{v}}^{n+1} \right) = \nabla \left(\nabla \cdot \widetilde{\mathbf{v}}^{n+1} \right) \quad \text{in } \Omega \tag{9}$$

$$\text{with} \quad \widehat{\mathbf{v}}^{n+1} \cdot \mathbf{n} = 0 \quad \text{on } \Gamma$$

$$\mathbf{v}^{n+1} = \widetilde{\mathbf{v}}^{n+1} + \widehat{\mathbf{v}}^{n+1}, \quad \text{and} \quad \nabla(p^{n+1} - p^n) = -\left(\frac{\rho^n}{\delta t} + \mu^n \mathbf{K}^{-1} \right) \widehat{\mathbf{v}}^{n+1} \quad \text{in } \Omega \tag{10}$$

This actually corresponds to the (VPP$_\varepsilon$) method proposed in [8, 10] which is also analyzed and studied to solve anisotropic and heterogeneous Darcy problems in [12].

The consistency of the (VPP$_\varepsilon$) method is ensured with (7) since we have using the fact that $\widehat{\mathbf{v}}^{n+1} = \mathbf{v}^{n+1} - \widetilde{\mathbf{v}}^{n+1}$:

$$\rho^n \frac{\mathbf{v}^{n+1} - \widetilde{\mathbf{v}}^{n+1}}{\delta t} + \nabla(p^{n+1} - p^n) = 0 \quad \text{with} \quad \mathbf{v}^{n+1} - \widetilde{\mathbf{v}}^{n+1} = \widehat{\mathbf{v}}^{n+1}. \tag{11}$$

Then, summing this last equation with the prediction step (5), we get the evolution equation satisfied by the velocity field $\mathbf{v}^{n+1}$ in Ω with the first-order linear implicit Euler scheme:

$$\rho^n \left(\frac{\mathbf{v}^{n+1} - \mathbf{v}^n}{\delta t} + (\mathbf{v}^n \cdot \nabla) \widetilde{\mathbf{v}}^{n+1} \right) - 2 \nabla \cdot \left(\mu^n \, \mathbf{d}(\widetilde{\mathbf{v}}^{n+1}) \right) + \mu^n \mathbf{K}^{-1} \widetilde{\mathbf{v}}^{n+1} + \nabla p^{n+1} = \mathbf{f}^n. \tag{12}$$

The key feature of our method is to calculate an accurate and curl-free approximation of the momentum vector correction $\rho^n \widehat{\mathbf{v}}^{n+1}$ in (6). Indeed (6-7) ensures that $\rho^n \widehat{\mathbf{v}}^{n+1}$ is exactly a gradient which justifies the choice for $\nabla \phi^{n+1} = \nabla(p^{n+1} - p^n)$ since we have:

$$\frac{\rho^n}{\delta t} \widehat{\mathbf{v}}^{n+1} = \frac{1}{\varepsilon} \nabla \left(\nabla \cdot \mathbf{v}^{n+1} \right) \quad \Rightarrow \quad \nabla(p^{n+1} - p^n) = -\frac{1}{\varepsilon} \nabla \left(\nabla \cdot \mathbf{v}^{n+1} \right). \tag{13}$$

From the mathematical point of view, since the domain Ω is connected and p^0 and $\nabla \cdot \mathbf{v}^{n+1}$ (with $\mathbf{v}^{n+1} \cdot \mathbf{n} = 0$ on Γ) have a null average in Ω with the divergence formula, $p^{n+1} \in L^2_0(\Omega)$ for all $n \in \mathbb{N}$ such that $(n+1)\delta t \leq T$ and the equivalence below holds:

$$\nabla(p^{n+1} - p^n) = -\frac{1}{\varepsilon} \nabla \left(\nabla \cdot \mathbf{v}^{n+1} \right) \quad \Leftrightarrow \quad p^{n+1} - p^n = -\frac{1}{\varepsilon} \nabla \cdot \mathbf{v}^{n+1}. \tag{14}$$

However, we generally do not use the above result for practical algorithms to avoid round-off errors when ε is too small; see Remark 5.

The (VPP$_\varepsilon$) method really takes advantage of the splitting method proposed in [9] for augmented Lagrangian systems or general saddle-point computations to get a very fast solution of (6); see [9, Theorem 1.1 and Corollary 1.2, 1.3].

When we need the pressure field itself p^{n+1}, e.g. to compute stress vectors at each time step, it is calculated in an incremental way as an auxiliary step. We propose to reconstruct $\phi^{n+1} = p^{n+1} - p^n$ from its gradient $\nabla\phi^{n+1}$ given in (7) as follows:

$$\begin{aligned}&\text{Auxiliary step:} \quad p^{n+1} = p^n + \phi^{n+1}\\ &\qquad \text{with } \phi^{n+1} \text{ reconstructed from its gradient} \quad \nabla\phi^{n+1} = -\frac{\rho^n}{\delta t}\,\widehat{\mathbf{v}}^{n+1} \quad \text{in } \Omega.\end{aligned} \tag{15}$$

Since this reconstruction is not at all necessary to run the numerical process, it is described below in Remark 2.

Remark 2 (Reconstruction of $\phi^{n+1} = (p^{n+1} - p^n)$ from its gradient if necessary.)

It is possible to reconstruct the discrete pressure field from its gradient calculated by the (VPP$_\varepsilon$) method with (15) on the meshed domain.

By circulating on a suitable path starting from a point on the boundary where $\phi^{n+1} = 0$ is fixed and going through all the pressure nodes in the mesh, we get with the gradient formula between two neighbour points A and B using the mid-point quadrature:

$$\phi^{n+1}(B) - \phi^{n+1}(A) = \int_A^B \nabla\phi^{n+1}\cdot d\mathbf{l} = -\int_A^B \frac{\rho^n}{\delta t}\,\widehat{\mathbf{v}}^{n+1}\cdot d\mathbf{l} \approx -\frac{\rho^n}{\delta t}\,|\widehat{\mathbf{v}}^{n+1}|\,h_{AB}, \tag{16}$$

where $h_{AB} = \text{distance}(A,B)$. The field ϕ^{n+1} is calculated point by point from the boundary and then passing successively by all the pressure nodes. Hence, this fast algorithm is performed at each time step to get the pressure field p^{n+1} from the known field p^n, if it is necessary.

The prediction step (5) being standard within splitting methods and its solution $\widetilde{\mathbf{v}}^{n+1}$ belonging to $H_0^1(\Omega)^d$, we now state below that the original vector (or velocity) penalty-projection step (6) is well-posed at each time step t_n. For that, we need the hypothesis below, assuming that the density remains uniformly bounded and that no vacuum appears:

$$(\mathscr{H}) \quad \rho^n \in L^\infty(\Omega) \ \text{ and } \ \exists\rho_{min} > 0,\ \rho^n(x) \geq \rho_{min} > 0 \text{ a.e. in } \Omega, \quad \forall n \in \mathbb{N},\, n\,\delta t \leq T. \tag{17}$$

Lemma 1 (Well-posedness of the velocity correction step (6).)

For all $\widetilde{\mathbf{v}}^{n+1}$ given in $H_{div}(\Omega)$, $\rho^n \in L^\infty(\Omega)$ satisfying the hypothesis $(\mathscr{H})$ in (17) and all $\varepsilon > 0$, $\delta t > 0$, there exists at each time step a unique solution $\widehat{\mathbf{v}}^{n+1}$ in $H_{div,0}(\Omega)$ to the velocity correction step (6). Moreover, $\sqrt{\rho^n}\,\widehat{\mathbf{v}}^{n+1} \in L^2(\Omega)^d$ and $\rho^n\,\widehat{\mathbf{v}}^{n+1}$ is curl-free: $\nabla\times(\rho^n\,\widehat{\mathbf{v}}^{n+1}) = 0$ a.e. in Ω. Then $\rho^n\,\widehat{\mathbf{v}}^{n+1}$ belongs to $H_{rot}(\Omega)$ and $\nabla\cdot(\widehat{\mathbf{v}}^{n+1} + \widetilde{\mathbf{v}}^{n+1})$ belongs to $H^1(\Omega)$ at each time step.

PROOF.

For all $\mathbf{u} \in H_{div}(\Omega)$, we recall the Green's formula defining the continuous normal trace $\mathbf{u}\cdot\mathbf{n}_{|\Gamma}$ in $H^{-\frac{1}{2}}(\Gamma)$, e.g. [40, Theorem 2.5], where $\langle .,. \rangle_{-1/2,\Gamma}$ denotes the duality pairing between $H^{-\frac{1}{2}}(\Gamma)$ and $H^{\frac{1}{2}}(\Gamma)$, and the resulting trace inequality:

$$\begin{aligned}&\int_\Omega \mathbf{u}\cdot\nabla\phi\,dx + \int_\Omega \nabla\cdot\mathbf{u}\,\phi\,dx = \langle \mathbf{u}\cdot\mathbf{n}, \phi\rangle_{-1/2,\Gamma}, \quad \text{for all } \phi \in H^1(\Omega)\\ &\|\mathbf{u}\cdot\mathbf{n}\|_{-\frac{1}{2},\Gamma} \leq \|\mathbf{u}\|_{Hdiv}, \quad \text{where} \quad \|\mathbf{u}\|_{Hdiv}^2 = \|\mathbf{u}\|_0^2 + \|\nabla\cdot\mathbf{u}\|_0^2.\end{aligned}$$

With $\varepsilon = \eta\, \delta t$, we define the bilinear form $a(.,.)$ in $H_{div,0}(\Omega) \times H_{div,0}(\Omega)$ and the linear form $l(.)$ in $H_{div,0}(\Omega)$ (for all $\widetilde{\mathbf{v}}^{n+1} \in H_{div}(\Omega)$) respectively by:

$$a(\mathbf{v},\mathbf{w}) = \eta \int_{\Omega} \rho^n\, \mathbf{v}\cdot\mathbf{w}\, dx + \int_{\Omega} \nabla\cdot\mathbf{v}\,\nabla\cdot\mathbf{w}\, dx, \quad \forall \mathbf{v},\mathbf{w} \in H_{div,0}(\Omega)$$
$$l(\mathbf{w}) = \int_{\Omega} \nabla\cdot\widetilde{\mathbf{v}}^{n+1}\,\nabla\cdot\mathbf{w}\, dx, \quad \forall \mathbf{w} \in H_{div,0}(\Omega).$$

Then, with the previous Green's formula and the boundary condition $\widehat{\mathbf{v}}^{n+1}\cdot\mathbf{n} = 0$ on Γ, it is clear that the weak form of (6) reads at each time step:

$$a(\widehat{\mathbf{v}}^{n+1},\mathbf{w}) = l(\mathbf{w}), \quad \forall \mathbf{w} \in H_{div,0}(\Omega).$$

With the assumption $(\mathscr{H})$ in (17), it is an easy matter to prove with the Lax-Milgram theorem, e.g. [57,21], that the weak problem above admits a unique solution $\widehat{\mathbf{v}}^{n+1}$ in the Hilbert space $H_{div,0}(\Omega)$ (as a closed subspace of $H_{div}(\Omega)$). Since ρ^n is bounded, $\rho^n\,\widehat{\mathbf{v}}^{n+1}$ belongs to $L^2(\Omega)^d$ at each time step.

Conversely, we now interpret the weak form to show that $\widehat{\mathbf{v}}^{n+1}$ is the strong solution of (6) in some sense. By taking a smooth and compactly supported test function $\mathbf{w} = \varphi$ in $\mathscr{C}_c^\infty(\Omega)^d = \mathscr{D}(\Omega)^d$, we get for all $\eta > 0$ in the distribution sense

$$\left\langle \eta\, \rho^n\, \widehat{\mathbf{v}}^{n+1} - \nabla\left(\nabla\cdot(\widehat{\mathbf{v}}^{n+1} + \widetilde{\mathbf{v}}^{n+1})\right), \varphi \right\rangle_{\mathscr{D}',\mathscr{D}} = 0, \quad \forall \varphi \in \mathscr{D}(\Omega)^d.$$

Since both $\nabla\cdot\widehat{\mathbf{v}}^{n+1}$ and $\nabla\cdot\widetilde{\mathbf{v}}^{n+1}$ belong to $L^2(\Omega)$ and thus having their gradients in $H^{-1}(\Omega)^d$, this implies that

$$\eta\, \rho^n\, \widehat{\mathbf{v}}^{n+1} - \nabla\left(\nabla\cdot\widehat{\mathbf{v}}^{n+1}\right) = \nabla\left(\nabla\cdot\widetilde{\mathbf{v}}^{n+1}\right) \quad \text{in } H^{-1}(\Omega)^d,$$

which means that $\widehat{\mathbf{v}}^{n+1} \in H_{div,0}(\Omega)$ is the strong solution to the velocity correction step (6) in the sense of $H^{-1}(\Omega)^d$. Since $\rho^n\,\widehat{\mathbf{v}}^{n+1}$ belongs to $L^2(\Omega)^d$, this equation is also satisfied in the sense of $L^2(\Omega)^d$ in the form below:

$$\eta\, \rho^n\, \widehat{\mathbf{v}}^{n+1} - \nabla\left(\nabla\cdot(\widehat{\mathbf{v}}^{n+1} + \widetilde{\mathbf{v}}^{n+1})\right) = 0 \quad \text{a.e. in } \Omega.$$

Moreover, this implies that $\nabla\cdot(\widehat{\mathbf{v}}^{n+1} + \widetilde{\mathbf{v}}^{n+1})$ belongs to $H^1(\Omega)$ and since $\rho^n\,\widehat{\mathbf{v}}^{n+1}$ is exactly a gradient, $\rho^n\,\widehat{\mathbf{v}}^{n+1}$ is curl-free. Thus, $\nabla\times(\rho^n\,\widehat{\mathbf{v}}^{n+1}) = 0$ in the sense of $H^{-1}(\Omega)^d$, and also in the sense of $L^2(\Omega)^d$, which completes the proof. □

Remark 3 (Regularization property of the penalty method.)

From Lemma 1, it comes that the velocity divergence $\nabla\cdot(\widehat{\mathbf{v}}^{n+1} + \widetilde{\mathbf{v}}^{n+1}) = \nabla\cdot\mathbf{v}^{n+1}$ belongs to $H^1(\Omega)\cap L^2_0(\Omega)$ at each time step. From the pressure correction (14), this implies that the pressure increment in time $\phi^{n+1} = p^{n+1} - p^n$ belongs to $H^1(\Omega)\cap L^2_0(\Omega)$ too, and thus gets an extra regularity. This is due to the regularizing effect of the penalty method which will be observed several times further in the paper.

Remark 4 (Connection with the original idea of Caltagirone.)

Taking formally the limit of the vector penalty-projection step (6) when $\varepsilon/\delta t$ tends to zero yields for a constant density ρ: $\nabla(\nabla\cdot\widehat{\mathbf{v}}^{n+1}) = -\nabla(\nabla\cdot\widetilde{\mathbf{v}}^{n+1})$ with $\mathbf{v}^{n+1} = \widetilde{\mathbf{v}}^{n+1} + \widehat{\mathbf{v}}^{n+1} \in H^1_n(\Omega)$. which corresponds to the original idea introduced in [25]. However, such a vector projection with $\varepsilon = 0$ is ill-posed since the div operator has not a null kernel and the grad(div) operator has many zero eigenvalues; see the numerical computation of grad(div) spectrum in e.g. [2] with spectral methods. This has led to the introduction of the vector penalty-projection methods proposed in [7]. Indeed, the singular vector correction step has a unique solution only with an additional constraint for $\widehat{\mathbf{v}}^{n+1}$, such that for example: $\nabla\times\widehat{\mathbf{v}}^{n+1}=0$ with $\widehat{\mathbf{v}}^{n+1}\cdot\mathbf{n}_{|\Gamma}=0$. This is what the present (VPP$_\varepsilon$) method effectively carries out by calculating, with a penalty method, a curl-free vector correction $\rho^n\,\widehat{\mathbf{v}}^{n+1}$ of the momentum; see Lemma 1.

2.2 Analogy and difference with other usual or less classical methods

2.2.1 Velocity penalty-projection and pressure gradient update versus projection methods

The (VPP$_\varepsilon$) splitting method uses a standard prediction step (5) which does not take the divergence-free constraint into account. The vector correction step (6-7) carries out an approximate divergence-free projection of the velocity, see Section 3, with the penalty parameter $\varepsilon>0$ chosen as small as desired. The time increment of pressure $p^{n+1}-p^n$ is never used to calculate $\mathbf{v}^{n+1}-\widetilde{\mathbf{v}}^{n+1}$. Moreover, the pressure field p^{n+1} is thus only updated by its gradient (7), and it can be reconstructed very fast from its gradient or simply calculated with (14) for ε not too small, only as a post-processing step. Besides, the method gets completely rid of explicit pressure boundary conditions. Those are fundamental differences from all projection or penalty-projection methods, both for incompressible flows [29,73,74,60,75,64,76, 62,45,44,47,63] or low Mach number dilatable flows [31,32,48], where a scalar correction step is performed by solving the homogeneous Neumann Poisson-like problem below for the pressure increment $\phi^{n+1}:=p^{n+1}-p^n$:

$$\nabla\cdot\left(\frac{\delta t}{\rho^n}\,\nabla\phi^{n+1}\right)=\nabla\cdot\widetilde{\mathbf{v}}^{n+1}\quad\text{ in }\Omega,\quad\text{ with }\ \nabla\phi^{n+1}\cdot\mathbf{n}_{|\Gamma}=0. \tag{18}$$

Indeed, the corresponding pressure update for the (VPP$_\varepsilon$) method amounts, at each time step, to the Neumann-Poisson problem below with the known velocity correction $\widehat{\mathbf{v}}^{n+1}$ (obtained by taking the divergence in Eq. (7)):

$$-\Delta\phi^{n+1}=\nabla\cdot\left(\frac{\rho^n}{\delta t}\,\widehat{\mathbf{v}}^{n+1}\right)\quad\text{ in }\Omega,\quad\text{ with }\ \nabla\phi^{n+1}\cdot\mathbf{n}_{|\Gamma}=0,\quad\phi^{n+1}=p^{n+1}-p^n. \tag{19}$$

However, the (VPP$_\varepsilon$) method does not require to solve (19) since the velocity and pressure gradient corrections are completely defined by the vector equations (6,7), despite it is always a pure Poisson problem even with variable density, which is not the case for the scalar projection methods. Very recently, a new scalar projection method was proposed in [46] for the variable density flow. However, this method seems restricted to weak mass density variations since only the case with a density ratio equal to 7 is computed. Besides, this method does not seem to be able to correctly compute the hydrostatic non-miscible diphasic case.

Furthermore, we would like to point out that the pressure correction (18) with the scalar projection methods involves a spatial derivative of the mass density. Conversely, the vector form of equations (6,7) for the (VPP$_\varepsilon$) method does not include any density spatial derivative, which is far more in agreement with the continuous equations (1,2).

2.2.2 Fast-(VPP) method versus scalar penalty-projection (SPP) method

In the scalar penalty-projection methods introduced in [47], an augmented Lagrangian term [39,41] with a parameter $r \geq 0$ is added in the prediction step whereas a consistent exact divergence-free projection is carried out; see also [68] where the case $r = 1/\delta t^2$ is analysed. In the first versions of VPP method [7], the main part of this augmentation term is splitted within the velocity correction step, which performs only an approximate divergence-free projection, since we have in fact:

$$r = r_0 + \frac{1}{\varepsilon}$$

where $r_0 \geq 10^{-4}$; see [9] and Remark 5. In the present VPP method, we have now completely eliminated the augmentation term from the prediction step, *i.e.* the method works well also with $r_0 = 0$. Indeed, we use the splitting augmented Lagrangian method recently proposed in [9] to get a fast solver for $\varepsilon = 1/r$ small enough where the right-hand side is adapted to the left-hand side operator in the correction step. Hence, although the correction step in the VPP is completely different than in the *SPP* where a consistent incremental projection method is used, see Section 2.2.1, the results of both methods can be compared. In particular, we refer to the *SPP* methods with sufficiently large values of r for vanishing the splitting errors [47,48], with theoretical analysis in [68,18,37], and for drastically reducing the spurious pressure boundary layer or yielding good convergence results in the case of open or outflow boundary conditions; see [47,48] for details.

2.2.3 Fast-(VPP) versus the first (VPP) method

By comparing the present fast-VPP with our first version of the VPP method [7], we notice that we have now completely eliminated both the diffusion and convection term in the vector correction step (6). This was in fact already suggested in [7] from the numerical point of view, but we can now justify it by the theoretical considerations in Section 3. By this way, we lose a little consistency, e.g. the fact that $\widehat{\mathbf{v}}^{n+1} \wedge \mathbf{n}_{|\Gamma} = 0$, but we gain back two crucial properties: $\rho^n \widehat{\mathbf{v}}^{n+1}$ *is exactly a gradient which is thus directly used for the pressure gradient update* and *the new correction step is far more faster and cheaper*. Indeed, the fast discrete solution to the present (VPP_ε) method is confirmed by the numerical results given in [9,10, 11]; see also Section 2.3.

Remark 5 (Pressure update.)

From the mathematical point of view, we can simply calculate the pressure with (14) as made in the first version of the VPP method [7]. However, we have observed that it is numerically far better to run the method by updating directly the pressure gradient (7) to avoid the effect of round-off errors when ε is very small. Indeed, the first VPP method requires an augmentation parameter $r_0 > 0$ in the prediction step to act as a preconditioner in order to get the first-order convergence of the pressure with the time step when the penalty parameter ε is very small.

2.2.4 High-order accuracy in time and open boundary conditions

The method can be written by using the Crank-Nicolson or 2nd-order backward finite difference (BDF2) schemes, instead of the Euler scheme, with suitable Richardson's extrapolations to get the second-order accuracy in time as it is usually made for other methods:

preconditioned fully coupled solver [16,30], Uzawa augmented Lagrangian [39,41,42] with finite elements or [26,50] with finite volumes on the MAC staggered mesh, scalar incremental projection [44] or scalar penalty-projection [47,48,37] methods. In order to naturally increase the time accuracy, the following evolution version of the (VPP$_\varepsilon$) method is also considered [14,15], where $\widetilde{\mathbf{v}}^0 = \mathbf{v}_0$ and $\widehat{\mathbf{v}}^0 = 0$ are given:

$$\rho^n \left(\frac{\widetilde{\mathbf{v}}^{n+1} - \widetilde{\mathbf{v}}^n}{\delta t} + (\mathbf{v}^n \cdot \nabla)\widetilde{\mathbf{v}}^{n+1} \right) - 2\nabla \cdot \left(\mu^n \, \mathbf{d}(\widetilde{\mathbf{v}}^{n+1}) \right) + \mu^n \, \mathbf{K}^{-1} \, \widetilde{\mathbf{v}}^{n+1} + \nabla p^n = \mathbf{f}^n \quad \text{in } \Omega \quad (20)$$

$$\text{with} \quad \widetilde{\mathbf{v}}^{n+1} = 0 \quad \text{on } \Gamma$$

$$\varepsilon \, \rho^n \, \frac{\widehat{\mathbf{v}}^{n+1} - \widehat{\mathbf{v}}^n}{\delta t} - \nabla \left(\nabla \cdot \widehat{\mathbf{v}}^{n+1} \right) = \nabla \left(\nabla \cdot \widetilde{\mathbf{v}}^{n+1} \right) \quad \text{in } \Omega \quad (21)$$

$$\text{with} \quad \widehat{\mathbf{v}}^{n+1} \cdot \mathbf{n} = 0 \quad \text{on } \Gamma$$

$$\mathbf{v}^{n+1} = \widetilde{\mathbf{v}}^{n+1} + \widehat{\mathbf{v}}^{n+1}, \quad \text{and} \quad \nabla (p^{n+1} - p^n) = -\rho^n \, \frac{\widehat{\mathbf{v}}^{n+1} - \widehat{\mathbf{v}}^n}{\delta t} \quad \text{in } \Omega. \quad (22)$$

By summing the prediction (20) and VPP-correction (21) steps using (22), it yields the same evolution equation (12) of the velocity field $\mathbf{v}^{n+1}$ as for the VPP version (5,6,7).

It is shown in [14,15] and see also [27] that the suitable second-order versions of the (VPP$_\varepsilon$) method effectively reach the second-order accuracy in time for both velocity and pressure, not only with a Dirichlet boundary condition for the velocity, but also for an open boundary condition with a given traction on a part of the border Γ. It is well-known, see e.g. [43,44] and the references therein, that this not at all the case with the scalar projection methods, even for the linear Stokes problem, except with the scalar penalty-projection method introduced and studied in [47,48,37,18].

2.2.5 A new two-step artificial compressibility method

By summing the prediction and correction steps (5–7), we get below the problem which is effectively satisfied by the discrete velocity $\mathbf{v}^{n+1}$ and pressure p^{n+1} and resulting from the proposed (VPP$_\varepsilon$) splitting method: see also the equations (12) and (14).

$$\rho^n \left(\frac{\mathbf{v}^{n+1} - \mathbf{v}^n}{\delta t} + (\mathbf{v}^n \cdot \nabla)\widetilde{\mathbf{v}}^{n+1} \right) - 2\nabla \cdot \left(\mu^n \, \mathbf{d}(\widetilde{\mathbf{v}}^{n+1}) \right) + \mu^n \, \mathbf{K}^{-1} \, \widetilde{\mathbf{v}}^{n+1} + \nabla p^{n+1} = \mathbf{f}^n \quad (23)$$

$$(\varepsilon \, \delta t) \, \frac{\nabla (p^{n+1} - p^n)}{\delta t} + \nabla \left(\nabla \cdot \mathbf{v}^{n+1} \right) = 0 \quad \Leftrightarrow \quad (\varepsilon \, \delta t) \, \frac{p^{n+1} - p^n}{\delta t} + \nabla \cdot \mathbf{v}^{n+1} = 0. \quad (24)$$

These equations are never solved in this form in the computational process. Nevertheless, they can be also viewed as defining a new two-step artificial or pseudo-compressibility method. Indeed, the previous method (23-24) differs from the original artificial compressibility method of Chorin-Temam [28,72] by three important features. This is a splitting method with a prediction-correction scheme which also works for non-homogeneous flows and the analogous continuous pressure equation reads here with an additional parameter $\xi > 0$:

$$\begin{aligned} (\varepsilon \, \xi) \, \partial_t \nabla p + \nabla \left(\nabla \cdot \mathbf{v} \right) &= 0 \quad \text{in } \Omega \times (0,T), \\ \text{or} \quad (\varepsilon \, \xi) \, \partial_t p + \nabla \cdot \mathbf{v} &= 0 \quad \text{in } \Omega \times (0,T), \end{aligned} \quad (25)$$

where we have $\xi = \delta t$ in the practical (VPP$_\varepsilon$) splitting method and $\xi = 1$ corresponds to the standard artificial compressibility method. The method (23,24) is also different from the pseudo-compressibility methods issued from projection methods; see [69,44] and Section 2.2.1 above.

The convergence analysis of such a continuous version of two-step artificial compressibility method in the case of a constant density is theoretically performed in [17] when $\varepsilon \to 0$. More precisely, by performing a compactness method using the Fourier transform, it is proved that the weak solutions of the compressibility method converge to weak solutions of the Navier-Stokes equations when $\varepsilon \to 0$ and whatever the fixed parameter $\xi > 0$.

In the case of low Mach number flows, typically when $M < 0.2$, it is possible to connect the parameter ε with the Mach number $M := V^\star/c$, $V^\star$ being a given reference velocity and c the velocity of acoustic waves; e.g. [23,24] :

$$c := \sqrt{\left(\frac{\partial p}{\partial \rho}\right)_S} = \sqrt{\frac{1}{\rho\,\chi_S}} = \sqrt{\frac{\gamma}{\rho\,\chi_\theta}},$$
$$\text{where} \quad \gamma := \frac{c_p}{c_v} \geq 1, \quad \chi_S := \frac{1}{\rho}\left(\frac{\partial \rho}{\partial p}\right)_S > 0, \quad \chi_\theta := \frac{1}{\rho}\left(\frac{\partial \rho}{\partial p}\right)_\theta = \gamma\chi_S > 0,$$

χ_S, χ_θ being respectively the isentropic and isothermal compressibility coefficients of the fluid. Now, from one hand we have with the continuity equation :

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = \frac{\partial \rho}{\partial t} + \mathbf{v}\cdot\nabla\rho = -\rho\,\nabla\cdot\mathbf{v}.$$

With a state law of the fluid like $\rho = \rho(p,\theta)$, a function of pressure and temperature, we have from another hand :

$$\begin{aligned}\frac{\mathrm{d}\rho}{\mathrm{d}t} &= \left(\frac{\partial \rho}{\partial p}\right)_\theta \frac{\mathrm{d}p}{\mathrm{d}t} + \left(\frac{\partial \rho}{\partial \theta}\right)_p \frac{\mathrm{d}\theta}{\mathrm{d}t} \\ &= \rho\,\chi_\theta\,\frac{\mathrm{d}p}{\mathrm{d}t} - \rho\,\beta\,\frac{\mathrm{d}\theta}{\mathrm{d}t}, \qquad \text{where} \quad \beta := -\frac{1}{\rho}\left(\frac{\partial \rho}{\partial \theta}\right)_p > 0\end{aligned}$$

is the coefficient of thermal volume expansion of the fluid. By combining the previous equalities, it yields the pressure equation below :

$$\chi_\theta\,\frac{\partial p}{\partial t} + \nabla\cdot\mathbf{v} = \beta\,\frac{\mathrm{d}\theta}{\mathrm{d}t} - \chi_\theta\,\mathbf{v}\cdot\nabla p.$$

Comparing this equation with (25) for the (VPP$_\varepsilon$) method, *i.e.* $\xi = \delta t$, in the case of a barotropic fluid ($\beta \approx 0$) or an isothermal flow at a constant temperature, and neglecting the last term in the right-hand side, we get

$$\varepsilon\,\delta t = \chi_\theta = \gamma\chi_S.$$

Hence, we find

$$M^2 := \frac{V^{\star 2}}{c^2} = \frac{\rho\,V^{\star 2}\,\chi_\theta}{\gamma} = \frac{\rho\,V^{\star 2}\,\varepsilon\,\delta t}{\gamma}, \quad \text{or also} \quad \gamma M^2 = \rho\,V^{\star 2}\,\varepsilon\,\delta t \ll 1. \tag{26}$$

2.3 On the fast discrete solution to the (VPP$_\varepsilon$) method

Let us now consider any space discretization of our problem. We denote by $B = -div_h$ the $m \times n$ matrix corresponding to the discrete divergence operator, $B^T = grad_h$ the $n \times m$ matrix corresponding to the discrete gradient operator, whereas I denotes the $n \times n$ identity matrix with $n > m$ and D the $n \times n$ diagonal nonsingular matrix containing all the discrete density values of $\rho^n > 0$ *a.e.* in Ω. Here n is the number of velocity unknowns whereas m is the number of pressure unknowns. Then, the discrete vector penalty-projection problem corresponding to (6) with $\varepsilon = \eta\, \delta t$ or (43) with $D = I$ reads:

$$\left(D + \frac{1}{\eta} B^T B\right) \widehat{v}_\eta = -\frac{1}{\eta} B^T B \widetilde{v}, \quad \text{with} \quad v_\eta = \widetilde{v} + \widehat{v}_\eta. \tag{27}$$

We proved in [9] a crucial result due to the *adapted right-hand side* in the correction step (27) which lies in the range of the limit operator $B^T B$. Indeed, (27) can be viewed as a singular perturbation problem with well-suited data in the right-hand side. More precisely, we state in [9, Theorem 1.1 and Corollary 1.3] the asymptotic expansion of the solution $\widehat{v}_\eta$ to (27):

$$\widehat{v}_\eta = -\frac{1}{\eta} \left(D + \frac{1}{\eta} B^T B\right)^{-1} B^T B \widetilde{v} \tag{28}$$

when the penalty parameter η is chosen sufficiently small. We also refer to [9, Corollary 1.2] for the very good effective conditioning of the whole linear system (27) when η is sufficiently small.

This explains why the solution can be obtained only within a few iterations of a suitable preconditioned conjugate gradient whatever the size of the mesh step or the dimension n. This is confirmed by the numerical results obtained in [9,10,11] with the *ILU(0)-BiCGStab2* preconditioned Krylov iterations or in [14,15] with the *IC(0)-PCG* preconditioned iterative solver.

3 Theoretical foundations of the vector penalty-projection

We present and analyze an *approximate divergence-free projection* problem and its solution using the splitting method proposed in [9] for general saddle-point problems. We also refer to [11] where fast Helmholtz-Hodge decompositions are proposed in bounded domains.

We first recall the Helmholtz-Hodge orthogonal decomposition of $L^2(\Omega)^d$ for a bounded domain, see [53,51,75]:

$$L^2(\Omega)^d = H \oplus \ H^\perp \quad \text{with:}$$
$$H = \{\mathbf{u} \in L^2(\Omega)^d;\ \nabla \cdot \mathbf{u} = 0,\ \mathbf{u} \cdot \mathbf{n} = 0 \text{ on } \Gamma\}, \quad H^\perp = \{\nabla \phi,\ \phi \in H^1(\Omega)\}.$$

Thus, for all $\widetilde{\mathbf{v}} \in L^2(\Omega)^d$, there exists a unique $(\mathbf{v}, q) \in H \times H^1(\Omega)/\mathbb{R}$ solution to the L^2 divergence-free projection:

$$\mathbf{v} + \nabla q = \widetilde{\mathbf{v}} \quad \text{with} \quad \nabla \cdot \mathbf{v} = 0 \quad \text{in } \Omega, \quad \text{and} \quad \mathbf{v} \cdot \mathbf{n} = 0 \quad \text{on } \Gamma. \tag{29}$$

This gives immediately the following bounds with Pythagore and the mean Poincaré inequality since $\int_\Omega q\, dx = 0$:

$$\|\mathbf{v}\|_0^2 + \|\nabla q\|_0^2 = \|\widetilde{\mathbf{v}}\|_0^2 \quad \text{and thus} \quad \|q\|_0 \le c_0(\Omega)\, \|\nabla q\|_0 \le c_0(\Omega)\, \|\widetilde{\mathbf{v}}\|_0. \tag{30}$$

If $\widetilde{\mathbf{v}}$ belongs to $H_{div,0}(\Omega)$, by writing $\Delta q = \nabla \cdot \widetilde{\mathbf{v}}$ with $\nabla q \cdot \mathbf{n}_{|\Gamma} = 0$, it is easy to show that the solution $q \in H^1(\Omega)/\mathbb{R}$ of this homogeneous Neumann-Poisson problem satisfies: $\|\nabla q\|_0 \leq \|\widetilde{\mathbf{v}}\|_0$. Moreover, if the domain Ω is convex, then q belongs to $H^2(\Omega)/\mathbb{R}$, see e.g. [1,36].

That leads to the very popular Chorin-Temam projection methods [29,73,74,60,75] and their many variants; see [44,31,32,47,48]. This decomposition is also at the basis of the family of SIMPLER methods introduced by Patankar and Spalding [59] which are very often used in computational fluid dynamics and heat or mass transfer.

3.1 The approximate penalty-projection (APP)

Now, we adopt a completely different point of view. The key idea of our method is to directly calculate an accurate curl-free approximation of the pressure gradient ∇q (the force inducing the motion) instead of determining the pressure itself (the Lagrange multiplier) to satisfy the exact velocity free-divergence constraint. We consider the solution of this problem with the penalty method, originally introduced by Courant [33] in a different context of constrained optimization to get problems with no constraint.

Let us study the following *approximate penalty-projection (APP)* problem for all $\eta > 0$ and all $\widetilde{\mathbf{v}}$ given in $L^2(\Omega)^d$, where we are looking for $(\mathbf{v}_\eta, q_\eta) \in H_{div,0}(\Omega) \times H^1(\Omega)/\mathbb{R}$:

$$(APP) \qquad \mathbf{v}_\eta + \nabla q_\eta = \widetilde{\mathbf{v}} \quad \text{with} \quad q_\eta = -\frac{1}{\eta}\nabla\cdot\mathbf{v}_\eta \quad \text{in } \Omega, \quad \text{and} \quad \mathbf{v}_\eta\cdot\mathbf{n} = 0 \quad \text{on } \Gamma$$

$$\Rightarrow \quad \mathbf{v}_\eta - \frac{1}{\eta}\nabla(\nabla\cdot\mathbf{v}_\eta) = \widetilde{\mathbf{v}} \quad \text{in } \Omega \quad \text{and} \quad \mathbf{v}_\eta\cdot\mathbf{n} = 0 \quad \text{on } \Gamma. \tag{31}$$

We prove the following optimal error estimates.

Theorem 1 (Approximate divergence-free penalty-projection.)
For all $\widetilde{\mathbf{v}}$ given in $L^2(\Omega)^d$ and $\eta > 0$, there exists a unique solution $\mathbf{v}_\eta$ in $H_{div,0}(\Omega)$ and q_η in $H^1(\Omega)\cap L^2_0(\Omega)$ to the approximate penalty-projection problem (31) and we have: $\nabla\times(\mathbf{v}_\eta - \mathbf{v}) = 0$ and then $(\mathbf{v}_\eta - \mathbf{v}) \in H^1_n(\Omega)$ and $(q_\eta - q) \in H^2(\Omega)$, where $(\mathbf{v}, q) \in H \times H^1(\Omega)/\mathbb{R}$ is the solution to (29) and satisfies (30). Moreover, there exists $c(\Omega) > 0$ such that the error estimate below holds:

$$\|\mathbf{v}_\eta - \mathbf{v}\|_1 + \|\nabla\cdot\mathbf{v}_\eta\|_0 + \|q_\eta - q\|_2 \leq c(\Omega)\,\|q\|_0\,\eta. \tag{32}$$

PROOF.

First, it is an easy matter to prove with the Lax-Milgram theorem that (31) admits a unique solution $\mathbf{v}_\eta$ in $H_{div,0}(\Omega)$: see for details the proof of Lemma 1 which considers a problem of the same type. Then we have $q_\eta = -\dfrac{1}{\eta}\nabla\cdot\mathbf{v}_\eta$ in $L^2_0(\Omega)$ since $\mathbf{v}_\eta\cdot\mathbf{n}_{|\Gamma} = 0$ and because $\nabla q_\eta = \widetilde{\mathbf{v}} - \mathbf{v}_\eta$ in $L^2(\Omega)^d$, we have also both q_η and $\nabla\cdot\mathbf{v}_\eta$ in $H^1(\Omega)$.

Second, from the difference between (31) and (29), we get since $\nabla\cdot\mathbf{v} = 0$:

$$\mathbf{v}_\eta - \mathbf{v} + \nabla(q_\eta - q) = 0 \quad \text{with} \quad q_\eta = -\frac{1}{\eta}\nabla\cdot\mathbf{v}_\eta = -\frac{1}{\eta}\nabla\cdot(\mathbf{v}_\eta - \mathbf{v}), \quad (\mathbf{v}_\eta - \mathbf{v})\cdot\mathbf{n}_{|\Gamma} = 0$$

and it yields the problem below satisfied by $\mathbf{v}_\eta - \mathbf{v}$ for all $\eta > 0$:

$$\mathbf{v}_\eta - \mathbf{v} - \frac{1}{\eta}\nabla(\nabla\cdot(\mathbf{v}_\eta - \mathbf{v})) = \nabla q \quad \text{in } \Omega \quad \text{and} \quad (\mathbf{v}_\eta - \mathbf{v})\cdot\mathbf{n} = 0 \quad \text{on } \Gamma. \tag{33}$$

Since $\mathbf{v}_\eta - \mathbf{v} = -\nabla(q_\eta - q)$, we have: $\nabla \times (\mathbf{v}_\eta - \mathbf{v}) = 0$, and as $\nabla \cdot (\mathbf{v}_\eta - \mathbf{v}) \in L^2(\Omega)$ with $(\mathbf{v}_\eta - \mathbf{v})\cdot \mathbf{n}_{|\Gamma} = 0$, then $\mathbf{v}_\eta - \mathbf{v}$ belongs to $H^1_n(\Omega)$, see [35,38,34]. Indeed, the space $H_{div,0}(\Omega) \cap H_{rot}(\Omega)$ is continuously imbedded in $H^1_n(\Omega)$, see also [40,3]. Then $q_\eta - q$ belongs to $H^2(\Omega)$ with $q \in H^1(\Omega)/\mathbb{R}$ and thus $q_\eta \in H^1(\Omega)/\mathbb{R}$. The fact that $(q_\eta - q) \in H^2(\Omega)$ is due to the regularization effect of the penalty method, see [40] and Remark 3.

Taking the L^2-inner product of (33) with $\mathbf{v}_\eta - \mathbf{v}$, we get using the Green's formula with $(\mathbf{v}_\eta - \mathbf{v})\cdot \mathbf{n}_{|\Gamma} = 0$, and then the Cauchy-Schwarz and Young inequalities:

$$\begin{aligned}\|\mathbf{v}_\eta - \mathbf{v}\|_0^2 + \frac{1}{\eta}\|\nabla\cdot(\mathbf{v}_\eta - \mathbf{v})\|_0^2 &= -\left(q, \nabla\cdot(\mathbf{v}_\eta - \mathbf{v})\right)_0 \\ &\le \frac{1}{2\eta}\|\nabla\cdot(\mathbf{v}_\eta - \mathbf{v})\|_0^2 + \frac{\eta}{2}\|q\|_0^2\end{aligned}$$

and thus:

$$\|\mathbf{v}_\eta - \mathbf{v}\|_0^2 + \frac{1}{2\eta}\|\nabla\cdot(\mathbf{v}_\eta - \mathbf{v})\|_0^2 \le \frac{\eta}{2}\|q\|_0^2 \quad \text{hence} \quad \|\nabla\cdot(\mathbf{v}_\eta - \mathbf{v})\|_0 \le \|q\|_0\,\eta. \tag{34}$$

Then, taking the L^2-inner product of (33) with $\nabla(q_\eta - q)$, we get since $(\mathbf{v}_\eta - \mathbf{v})\cdot \mathbf{n}_{|\Gamma} = 0$ and using the mean Poincaré inequality [57,21] with $\int_\Omega (q_\eta - q)\,dx = 0$:

$$\begin{aligned}\|\mathbf{v}_\eta - \mathbf{v}\|_0 = \|\nabla(q_\eta - q)\|_0 &\le c_0(\Omega)\,\|q\|_0\,\eta \\ \|q_\eta - q\|_0 \le c_0(\Omega)\,\|\nabla(q_\eta - q)\|_0 &\le c_0(\Omega)^2\,\|q\|_0\,\eta.\end{aligned} \tag{35}$$

Since $(\mathbf{v}_\eta - \mathbf{v})\cdot \mathbf{n}_{|\Gamma} = 0$, we now observe that the H^1-norm $\|\mathbf{v}_\eta - \mathbf{v}\|_1$ is equivalent to the norm $(\|\mathbf{v}_\eta - \mathbf{v}\|_0 + \|div(\mathbf{v}_\eta - \mathbf{v})\|_0 + \|curl(\mathbf{v}_\eta - \mathbf{v})\|_0)$, see [35,38,40] or [3] for a complete review. Thus with (34) and (35), since $\nabla \times (\mathbf{v}_\eta - \mathbf{v}) = 0$, there exists $c_1(\Omega) > 0$ such that:

$$\begin{aligned}\|\nabla(q_\eta - q)\|_1 = \|\mathbf{v}_\eta - \mathbf{v}\|_1 &\le c_1(\Omega)\,(\|\mathbf{v}_\eta - \mathbf{v}\|_0 + \|\nabla\cdot(\mathbf{v}_\eta - \mathbf{v})\|_0) \\ &\le c_1(\Omega)\,(1 + c_0(\Omega))\,\|q\|_0\,\eta = c(\Omega)\,\|q\|_0\,\eta.\end{aligned} \tag{36}$$

This concludes the proof of (32) with the previous estimates (34,35,36). □

Then, using a similar idea of Temam [75] for the Stokes problem, we can refine the previous result with an asymptotic expansion of $(\mathbf{v}_\eta, q_\eta)$ in powers of η. More precisely, we define by induction a sequence $(\mathbf{v}_k, q_k) \in H_{div,0}(\Omega) \times H^1(\Omega)/\mathbb{R}$ as the solution of the following problem for any integer $k \ge 1$:

$$\begin{aligned}\mathbf{v}_k + \nabla q_k = 0 \quad \text{and} \quad & \nabla\cdot\mathbf{v}_k = -q_{k-1}, \quad \text{where} \quad \mathbf{v}_0 = \mathbf{v},\ q_0 = q \\ \Rightarrow \quad & \Delta q_k = q_{k-1} \quad \text{with} \quad \nabla q_k \cdot \mathbf{n}_{|\Gamma} = 0.\end{aligned} \tag{37}$$

It is easy to show, knowing q_{k-1}, that (37) defines a unique pair $(\mathbf{v}_k, q_k) \in H_{div,0}(\Omega) \times H^1(\Omega)/\mathbb{R}$ for all $k \in \mathbb{N}$. Then, we have the following result by defining below the errors of $(\mathbf{v}_\eta, q_\eta)$ from the asymptotic series at any order $N \in \mathbb{N}$:

$$\mathbf{w}_\eta^N = \mathbf{v}_\eta - \mathbf{v} - \sum_{k=1}^{N} \mathbf{v}_k\,\eta^k \quad \text{and} \quad \lambda_\eta^N = q_\eta - q - \sum_{k=1}^{N} q_k\,\eta^k, \qquad \forall N \in \mathbb{N}. \tag{38}$$

Theorem 2 (Error estimates with asymptotic expansion.)
The sequence $(\mathbf{v}_k, q_k)$ *solution to (37) for* $k \geq 1$ *verifies:* $\nabla \times \mathbf{v}_k = 0$ *and* $(\mathbf{v}_k, q_k)$ *belongs to* $H^1_n(\Omega) \times H^2(\Omega)/\mathbb{R}$*. Moreover, for any* $N \in \mathbb{N}$*, there exists* $c_0(\Omega), c(\Omega) > 0$ *such that,* $\forall \widetilde{\mathbf{v}} \in L^2(\Omega)^d$ *and* $\forall \eta > 0$*, the error (38) of the approximate penalty-projection solution to (31) satisfies:*

$$\begin{aligned} \|\mathbf{w}^N_\eta\|_1 = \|\nabla \lambda^N_\eta\|_1 &\leq c(\Omega)\, c_0(\Omega)^{2N}\, \|q\|_0\, \eta^{N+1} \\ \|\lambda^N_\eta\|_0 &\leq c_0(\Omega)^{2N+2}\, \|q\|_0\, \eta^{N+1} \\ \|\nabla \cdot \mathbf{w}^N_\eta\|_0 &\leq c_0(\Omega)^{2N}\, \|q\|_0\, \eta^{N+1}. \end{aligned} \tag{39}$$

PROOF.

The proof is similar to that of Theorem 1 where the estimates (34,35,36) correspond to the case $N = 0$.

First from (37), we have for all $k \geq 1$: $\nabla \times \mathbf{v}_k = 0$, and, since $\nabla \cdot \mathbf{v}_k \in L^2(\Omega)$ with $\mathbf{v}_k \cdot \mathbf{n}_{|\Gamma} = 0$, then $\mathbf{v}_k$ belongs to $H^1_n(\Omega)$ and thus q_k belongs to $H^2(\Omega)/\mathbb{R}$. We also get by an easy induction using the mean Poincaré inequality that:

$$\|q_k\|_0 \leq c_0(\Omega)^{2k}\, \|q\|_0 \quad \text{and} \quad \|\nabla q_k\|_0 \leq c_0(\Omega)^{2k}\, \|\nabla q\|_0, \qquad \forall k \in \mathbb{N}.$$

Now, an easy calculation from (31,29) with (38) and (37) yields that $(\mathbf{w}^N_\eta, \lambda^N_\eta)$ is the solution to the problem below:

$$\mathbf{w}^N_\eta + \nabla \lambda^N_\eta = 0 \quad \text{and} \quad \nabla \cdot \mathbf{w}^N_\eta = -q_N\, \eta^{N+1} \quad \Rightarrow \quad \Delta \lambda^N_\eta = q_N\, \eta^{N+1} \quad \text{with } \nabla \lambda^N_\eta \cdot \mathbf{n}_{|\Gamma} = 0.$$

Then, we get with the previous bound:

$$\|\nabla \cdot \mathbf{w}^N_\eta\|_0 \leq \|q_N\|_0\, \eta^{N+1} \leq c_0(\Omega)^{2N}\, \|q\|_0\, \eta^{N+1}. \tag{40}$$

Since $\mathbf{w}^N_\eta = -\nabla \lambda^N_\eta$, we have: $\nabla \times \mathbf{w}^N_\eta = 0$, and as $\nabla \cdot \mathbf{w}^N_\eta \in L^2(\Omega)$ with $\mathbf{w}^N_\eta \cdot \mathbf{n}_{|\Gamma} = 0$, then $\mathbf{w}^N_\eta$ belongs to $H^1_n(\Omega)$, see [38]. Then λ^N_η belongs to $H^2(\Omega)/\mathbb{R}$. Moreover, using the mean Poincaré inequality, we get similarly to (35):

$$\begin{aligned} \|\mathbf{w}^N_\eta\|_0 = \|\nabla \lambda^N_\eta\|_0 &\leq c_0(\Omega)^{2N+1}\, \|q\|_0\, \eta^{N+1} \\ \|\lambda^N_\eta\|_0 \leq c_0(\Omega)\, \|\nabla \lambda^N_\eta\|_0 &\leq c_0(\Omega)^{2N+2}\, \|q\|_0\, \eta^{N+1}. \end{aligned} \tag{41}$$

Besides, similarly to (36) with the equivalent H^1 norm of $\mathbf{w}^N_\eta$ since $\mathbf{w}^N_\eta \cdot \mathbf{n}_{|\Gamma} = 0$ and $\nabla \times \mathbf{w}^N_\eta = 0$, we have:

$$\begin{aligned} \|\nabla \lambda^N_\eta\|_1 = \|\mathbf{w}^N_\eta\|_1 &\leq c_1(\Omega) \left(\|\mathbf{w}^N_\eta\|_0 + \|\nabla \cdot \mathbf{w}^N_\eta\|_0 \right) \\ &\leq c(\Omega)\, c_0(\Omega)^{2N}\, \|q\|_0\, \eta^{N+1}. \end{aligned} \tag{42}$$

This completes the proof with the previous estimates. □

3.2 The vector penalty-projection (VPP)

Since the *(APP)* problem (31) is ill-conditioned when η is small, we now study an efficient splitting method to solve (31) as proposed in [9] for general saddle-point problems. Here, it amounts to directly seek the curl-free vector correction $\widehat{\mathbf{v}}_\eta = -\nabla q_\eta$ such that $\mathbf{v}_\eta = \widetilde{\mathbf{v}} + \widehat{\mathbf{v}}_\eta$ be the solution of (31), which requires a more regular data $\widetilde{\mathbf{v}}$. Thus, for any $\widetilde{\mathbf{v}}$ given in $H_{div}(\Omega)$, we consider the so-called *vector penalty-projection (VPP)* problem below for all $\eta > 0$:

$$(VPP) \qquad \widehat{\mathbf{v}}_\eta - \frac{1}{\eta}\nabla(\nabla\cdot\widehat{\mathbf{v}}_\eta) = \frac{1}{\eta}\nabla(\nabla\cdot\widetilde{\mathbf{v}}) \qquad \text{with } \mathbf{v}_\eta = \widetilde{\mathbf{v}} + \widehat{\mathbf{v}}_\eta, \quad q_\eta = -\frac{1}{\eta}\nabla\cdot\mathbf{v}_\eta \tag{43}$$

$$\Rightarrow \quad \widehat{\mathbf{v}}_\eta = \frac{1}{\eta}\nabla(\nabla\cdot\mathbf{v}_\eta), \qquad \text{for all } \eta > 0.$$

This problem is well-posed in $H_{div,0}(\Omega)$, see Lemma 2 below. Moreover, for all $\widetilde{\mathbf{v}} \in H_{div,0}(\Omega)$, the problems (31) and (43) are equivalent and thus the error estimates in Theorems 1 and 2 hold. This actually gives the proof of [11, Theorem 3.3] which was just stated there without proof for sake of shortness. Let us also notice that (43) corresponds to the vector correction step (6) performed at each time step in the proposed (VPP$_\varepsilon$) method with $\varepsilon = \eta\,\delta t$, whereas $\widetilde{\mathbf{v}}$ is calculated by a prediction step which does not take the divergence-free constraint into account.

Lemma 2 (Well-posedness of the vector penalty-projection (VPP).)
For any $\widetilde{\mathbf{v}}$ given in $H_{div}(\Omega)$ and all $\eta > 0$, there exists at each time step a unique solution $\widehat{\mathbf{v}}_\eta$ in $H_{div,0}(\Omega)$ to the vector penalty-projection (43), i.e. *also to the velocity correction step (6) with $\varepsilon = \eta\,\delta t$ for a constant density $\rho > 0$. Moreover, $\widehat{\mathbf{v}}_\eta$ is curl-free: $\nabla\times\widehat{\mathbf{v}}_\eta = 0$ and since $\widehat{\mathbf{v}}_\eta\cdot\mathbf{n}_{|\Gamma} = 0$, then we have $\widehat{\mathbf{v}}_\eta \in H^1_n(\Omega)$ and satisfies the bound below with $c_1(\Omega) > 0$:*

$$\|\widehat{\mathbf{v}}_\eta\|_1 = \|\nabla q_\eta\|_1 \le c_1(\Omega)\left(\|\widehat{\mathbf{v}}_\eta\|_0 + \|\nabla\cdot\widehat{\mathbf{v}}_\eta\|_0\right) \le c_1(\Omega)\left(2\|\widetilde{\mathbf{v}}\|_0 + \|\nabla\cdot\widetilde{\mathbf{v}}\|_0\right), \quad \forall\eta > 0.$$

PROOF.
By using the previous Lemma 1, there exists a unique solution $\widehat{\mathbf{v}}_\eta$ in $H_{div,0}(\Omega)$ to (43) and we have: $\nabla\times\widehat{\mathbf{v}}_\eta = 0$ since $\mathbf{v}_\eta = \widetilde{\mathbf{v}} + \widehat{\mathbf{v}}_\eta$ and thus $\widehat{\mathbf{v}}_\eta$ is exactly a gradient for all $\eta > 0$. Since $\widehat{\mathbf{v}}_\eta$ in $H_{div}(\Omega)$ is curl-free with $\widehat{\mathbf{v}}_\eta\cdot\mathbf{n}_{|\Gamma} = 0$, then $\widehat{\mathbf{v}}_\eta$ belongs to $H^1_n(\Omega)$, see [38]. Besides, with the equivalence of the H^1-norm to the norm $(\|.\|_0 + \|div.\|_0 + \|curl.\|_0)$ for functions in $H^1_n(\Omega)$, see [35,38,40], there exists $c_1(\Omega)$ such that:
$\|\widehat{\mathbf{v}}_\eta\|_1 \le c_1(\Omega)\left(\|\widehat{\mathbf{v}}_\eta\|_0 + \|\nabla\cdot\widehat{\mathbf{v}}_\eta\|_0\right)$. Then, a simple energy estimate with (43) gives:

$$\|\widehat{\mathbf{v}}_\eta\|_0^2 + \frac{1}{2\eta}\|\nabla\cdot\widehat{\mathbf{v}}_\eta\|_0^2 \le \frac{1}{2\eta}\|\nabla\cdot\widetilde{\mathbf{v}}\|_0^2.$$

This implies that:

$$\|\nabla\cdot\widehat{\mathbf{v}}_\eta\|_0 \le \|\nabla\cdot\widetilde{\mathbf{v}}\|_0, \qquad \text{for all } \eta > 0.$$

Furthermore, by a simple energy estimate with (31), we get:

$$\frac{1}{2}\|\mathbf{v}_\eta\|_0^2 + \frac{1}{\eta}\|\nabla\cdot\mathbf{v}_\eta\|_0^2 \le \frac{1}{2}\|\widetilde{\mathbf{v}}\|_0^2.$$

Thus we have $\|\mathbf{v}_\eta\|_0 \le \|\widetilde{\mathbf{v}}\|_0$, and it also gives with the triangular inequality:

$$\|\widehat{\mathbf{v}}_\eta\|_0 \le \|\widetilde{\mathbf{v}}\|_0 + \|\mathbf{v}_\eta\|_0 \le 2\|\widetilde{\mathbf{v}}\|_0, \qquad \text{for all } \eta > 0.$$

This finally yields the desired bound for $\|\widehat{\mathbf{v}}_\eta\|_1$ with the previous estimates. □

The great interest of solving (43) instead of (31) is explained in Section 2.3 issued from [8,9] which shows that the method can be ultra-fast and very cheap if η is sufficiently small since (43) includes a right-hand side which is *adapted to the left-hand side operator*. This is also confirmed by the numerical results given in [10,11].

The present section fully justifies the name of the method as a vector penalty-projection. Moreover, we have shown that the velocity correction step (6) in the (VPP$_\varepsilon$) splitting method carries out an approximate divergence-free projection with a penalty method which yields at each time step t_n a divergence error for the velocity of order $\mathscr{O}(\varepsilon/\delta t)$ at least.

4 Analysis of the (VPP$_\varepsilon$) method for homogeneous flows

4.1 Preliminary on the nonlinear convection term

To deal with the nonlinear convection term, we use the trilinear skew-symmetric form $b(.,.,.)$ introduced by Temam [72,75] and defined by:
for all $\mathbf{u} \in H^1_n(\Omega)$ and $\mathbf{v}$, $\mathbf{w} \in H^1(\Omega)^d$,

$$\begin{aligned} b(\mathbf{u},\mathbf{v},\mathbf{w}) &= \int_\Omega ((\mathbf{u}\cdot\nabla)\mathbf{v})\cdot\mathbf{w}\,dx + \frac{1}{2}\int_\Omega (\nabla\cdot\mathbf{u})\,\mathbf{v}\cdot\mathbf{w}\,dx \\ &= \frac{1}{2}\int_\Omega ((\mathbf{u}\cdot\nabla)\mathbf{v})\cdot\mathbf{w}\,dx - \frac{1}{2}\int_\Omega ((\mathbf{u}\cdot\nabla)\mathbf{w})\cdot\mathbf{v}\,dx. \end{aligned} \tag{44}$$

The corresponding bilinear operator $B(.,.)$ is defined on $H^1_n(\Omega)\times H^1(\Omega)^d$ as below:

$$B(\mathbf{u},\mathbf{v}) = (\mathbf{u}\cdot\nabla)\mathbf{v} + \frac{1}{2}(\nabla\cdot\mathbf{u})\,\mathbf{v}. \tag{45}$$

Moreover, the following proposition holds.

Proposition 1 (Skew-symmetry property.)
The trilinear form $b(.,.,.)$ defined in (44) satisfies for all $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ in $H^1(\Omega)^d$:

$$b(\mathbf{u},\mathbf{v},\mathbf{w}) = -b(\mathbf{u},\mathbf{w},\mathbf{v}) + \sum_{i,j=1}^{d}\int_\Gamma (\mathbf{u}_i\,\mathbf{n}_i)\,\mathbf{v}_j\,\mathbf{w}_j\,ds.$$

PROOF.
Using the usual convention of implicit summation on the repeated index, we have:

$$\begin{aligned} b(\mathbf{u},\mathbf{v},\mathbf{w}) &= \int_\Omega \mathbf{u}_i\,\frac{\partial \mathbf{v}_j}{\partial x_i}\,\mathbf{w}_j\,dx + \frac{1}{2}\int_\Omega \frac{\partial \mathbf{u}_i}{\partial x_i}\,\mathbf{v}_j\,\mathbf{w}_j\,dx \\ &= \int_\Omega \frac{\partial(\mathbf{u}_i\,\mathbf{v}_j)}{\partial x_i}\,\mathbf{w}_j\,dx - \int_\Omega \frac{\partial \mathbf{u}_i}{\partial x_i}\,\mathbf{v}_j\,\mathbf{w}_j\,dx + \frac{1}{2}\int_\Omega \frac{\partial \mathbf{u}_i}{\partial x_i}\,\mathbf{v}_j\,\mathbf{w}_j\,dx \\ &= \int_\Omega \frac{\partial(\mathbf{u}_i\,\mathbf{v}_j)}{\partial x_i}\,\mathbf{w}_j\,dx - \frac{1}{2}\int_\Omega \frac{\partial \mathbf{u}_i}{\partial x_i}\,\mathbf{v}_j\,\mathbf{w}_j\,dx. \end{aligned}$$

Then, applying the Green formula on the first term in the right-hand side, we get the desired result:

$$b(\mathbf{u},\mathbf{v},\mathbf{w}) = -\int_\Omega \mathbf{u}_i\,\mathbf{v}_j\,\frac{\partial \mathbf{w}_j}{\partial x_i}\,dx - \frac{1}{2}\int_\Omega \frac{\partial \mathbf{u}_i}{\partial x_i}\,\mathbf{v}_j\,\mathbf{w}_j\,dx + \int_\Gamma (\mathbf{u}_i\,\mathbf{n}_i)\,\mathbf{v}_j\,\mathbf{w}_j\,ds.$$

□

Thus, we have from the previous Proposition 1 the following skew-symmetry property of $b(.,.,.)$ with respect to the two last arguments:

$$b(\mathbf{u},\mathbf{v},\mathbf{w}) = -b(\mathbf{u},\mathbf{w},\mathbf{v}), \quad b(\mathbf{u},\mathbf{v},\mathbf{v}) = 0, \quad \forall \mathbf{u}\in H^1_n(\Omega),\ \forall \mathbf{v},\,\mathbf{w}\in H^1(\Omega)^d. \tag{46}$$

4.2 Stability analysis of the (VPP$_\varepsilon$) method for homogeneous Navier-Stokes flows

In the case of Navier-Stokes problems for incompressible flows with a constant density $\rho > 0$ and viscosity $\mu > 0$, we use the dimensionless equations with $Re := \rho V^\star L^\star/\mu$ denoting the Reynolds number and $Da := K^\star/L^{\star 2}$ the Darcy number, where the quantities with a star index are some chosen reference values; see [50]. Hence, the prediction (5) and correction steps (6,7) now read:

$$\frac{\widetilde{\mathbf{v}}^{n+1} - \mathbf{v}^n}{\delta t} + B\left(\mathbf{v}^n, \widetilde{\mathbf{v}}^{n+1}\right) - \frac{1}{Re}\Delta\widetilde{\mathbf{v}}^{n+1} + \frac{1}{Re\,Da}\mathbf{K}^{-1}\widetilde{\mathbf{v}}^{n+1} + \nabla p^n = \mathbf{f}^{n+1} \qquad \text{in } \Omega \quad (47)$$

$$\frac{\varepsilon}{\delta t}\widehat{\mathbf{v}}^{n+1} - \nabla\left(\nabla\cdot\widehat{\mathbf{v}}^{n+1}\right) = \nabla\left(\nabla\cdot\widetilde{\mathbf{v}}^{n+1}\right) \qquad \text{in } \Omega \quad (48)$$

$$\mathbf{v}^{n+1} = \widetilde{\mathbf{v}}^{n+1} + \widehat{\mathbf{v}}^{n+1}, \quad \text{and} \quad \frac{\mathbf{v}^{n+1} - \widetilde{\mathbf{v}}^{n+1}}{\delta t} + \nabla(p^{n+1} - p^n) = 0 \qquad \text{in } \Omega \quad (49)$$

with $\widetilde{\mathbf{v}}^{n+1}_{|\Gamma} = 0$ and $\widehat{\mathbf{v}}^{n+1}\cdot\mathbf{n}_{|\Gamma} = 0$.

Then, we prove the well-posedness of the (VPP$_\varepsilon$) method (47-49).

Lemma 3 (Global solvability of (VPP$_\varepsilon$) method for homogeneous Navier-Stokes.)
For $\mathbf{f} \in L^2(0,T;H^{-1}(\Omega)^d)$, $\mathbf{v}^0 \in L^2(\Omega)^d$ *and* $p^0 \in L^2_0(\Omega)$ *given, the (VPP$_\varepsilon$) method is well-posed for all* $0 < \delta t \le T$ *and* $\varepsilon > 0$, i.e. *for all* $n \in \mathbb{N}$ *such that* $(n+1)\,\delta t \le T$, *there exists a unique solution* $(\widetilde{\mathbf{v}}^{n+1}, \mathbf{v}^{n+1}, p^{n+1}) \in H^1_0(\Omega)^d \times H^1_n(\Omega) \times L^2_0(\Omega)$ *to the (VPP$_\varepsilon$) scheme (47-49) such that:*

$$\frac{\mathbf{v}^{n+1} - \mathbf{v}^n}{\delta t} + B\left(\mathbf{v}^n, \widetilde{\mathbf{v}}^{n+1}\right) - \frac{1}{Re}\Delta\widetilde{\mathbf{v}}^{n+1} + \frac{1}{Re\,Da}\mathbf{K}^{-1}\widetilde{\mathbf{v}}^{n+1} + \nabla p^{n+1} = \mathbf{f}^{n+1} \qquad \textit{in } \Omega \quad (50)$$

$$(\varepsilon\,\delta t)\,\frac{p^{n+1} - p^n}{\delta t} + \nabla\cdot\mathbf{v}^{n+1} = 0 \qquad \textit{in } \Omega \quad (51)$$

which is the discrete problem effectively solved by this two-step artificial compressibility splitting scheme.

Moreover, we have both $\nabla\cdot\mathbf{v}^{n+1}$ *and* $\phi^{n+1} := p^{n+1} - p^n$ *belonging to* $H^2(\Omega)\cap L^2_0(\Omega)$ *and hence* $p^{n+1} \in H^2(\Omega)\cap L^2_0(\Omega)$ *if the initial pressure* p^0 *is chosen in* $H^2(\Omega)\cap L^2_0(\Omega)$.

PROOF.
The proof is made by induction for all $n \in \mathbb{N}$ such that $(n+1)\,\delta t \le T$ from the given initial conditions $\mathbf{v}^0 \in L^2(\Omega)^d$ and $p^0 \in L^2_0(\Omega)$.

At each time step t_{n+1}, there exists a unique solution $\widetilde{\mathbf{v}}^{n+1} \in H^1_0(\Omega)^d$ to the prediction step (47). Indeed, it is an easy application of the Lax-Milgram theorem for this linear advection-diffusion problem where $b(\mathbf{v}^n, \widetilde{\mathbf{v}}^{n+1}, \mathbf{w}) = 0$ for all $\mathbf{w} \in H^1_0(\Omega)^d$ from Proposition 1, which ensures the coercivity of this problem in $H^1_0(\Omega)^d$.

Moreover, from Lemma 2, there exists a unique solution $\widehat{\mathbf{v}}^{n+1} \in H^1_n(\Omega)$ to the velocity correction step (48). Thus, we have $\mathbf{v}^{n+1} = \widetilde{\mathbf{v}}^{n+1} + \widehat{\mathbf{v}}^{n+1}$ which belongs to $H^1_n(\Omega)$. From another hand with (14), since both p^0 and $\nabla\cdot\mathbf{v}^{n+1}$ belong to $L^2_0(\Omega)$ and the domain Ω is connected, then p^{n+1} belongs to $L^2_0(\Omega)$ and satisfies (51).

Hence, the well-posedness of the whole (VPP$_\varepsilon$) scheme (47-49) follows by an immediate induction.

Furthermore, the additional regularity of the velocity divergence and pressure is brought by the regularizing effect of the penalty method, see Remark 3, and with (48) by the continuous imbedding of the space $H_{div,0}(\Omega)\cap H_{rot}(\Omega)$ in $H^1_n(\Omega)$, see e.g. [40,3], and thus $\widehat{\mathbf{v}}^{n+1} \in H^1_n(\Omega)$ from Lemma 2. This concludes the proof. □

We now prove below the unconditional stability of the (VPP$_\varepsilon$) method. Here we omit the Darcy's drag term in (47) since it brings no additional complication; see [12] for the study of Darcy problems.

Theorem 3 (Stability of (VPP$_\varepsilon$) method for homogeneous Navier-Stokes problems.)

For $\mathbf{f} \in L^2(0,T;H^{-1}(\Omega)^d)$, $\mathbf{v}^0 \in L^2(\Omega)^d$ *and* $p^0 \in L^2_0(\Omega) \cap H^1(\Omega)$, *there exists* $C_j > 0$ *for* $j = 0, 2$ *with* $C_j = C_j\left(\Omega, T, Re, \|\mathbf{f}\|_{L^2(0,T;H^{-1})}, \|\mathbf{v}^0\|_0, \|p^0\|_1\right)$ *such that, for all* $0 < \varepsilon \le 1$ *and* $0 < \delta t \le T$, *the solution* $(\widetilde{\mathbf{v}}^{n+1}, \mathbf{v}^{n+1}, p^{n+1})$ *of the (VPP$_\varepsilon$) method (47-49) satisfies for all* $n \in \mathbb{N}$ *with* $(n+1)\,\delta t \le T$ *the following estimates where* $c_0(\Omega)$ *denotes the mean Poincaré constant:*

$$(i) \qquad \|\mathbf{v}^{n+1}\|_0^2 + \sum_{k=0}^{n} \|\widetilde{\mathbf{v}}^{k+1} - \mathbf{v}^k\|_0^2 + \frac{1}{Re}\sum_{k=0}^{n} \delta t\, \|\nabla \widetilde{\mathbf{v}}^{k+1}\|_0^2$$

$$+\delta t^2\, \|\nabla p^{n+1}\|_0^2 + \varepsilon\, \delta t\, \|p^{n+1}\|_0^2 + \varepsilon \sum_{k=0}^{n} \delta t\, \|p^{k+1} - p^k\|_0^2 \le C_0$$

$$(ii) \qquad \sum_{k=0}^{n} \delta t\, \|\nabla \cdot \mathbf{v}^{k+1}\|_0^2 \le C_0\, \varepsilon$$

$$(iii) \qquad \|\pi^{n+1}\|_1^2 \le \left(1 + c_0(\Omega)^2\right) C_0 \quad \textit{with} \quad \pi^{n+1} = \delta t\, p^{n+1}$$

$$(iv_a) \qquad \sum_{k=0}^{n} \delta t\, \|\nabla \mathbf{v}^{k+1}\|_0^2 + \sum_{k=0}^{n} \delta t\, \|\nabla(\mathbf{v}^{k+1} - \widetilde{\mathbf{v}}^{k+1})\|_0^2 \le C_1, \quad \textit{if } \Omega \textit{ simply connected.}$$

Moreover, if we assume that: $\varepsilon = \lambda\, \delta t$ *with a given constant* $\lambda > 0$, *we have the additional bounds below including the time increments of velocity which hold without the assumption that* Ω *is simply connected:*

$$(iv_b) \qquad \sum_{k=0}^{n} \delta t\, \|\nabla \mathbf{v}^{k+1}\|_0^2 + \sum_{k=0}^{n} \delta t\, \|\nabla(\mathbf{v}^{k+1} - \widetilde{\mathbf{v}}^{k+1})\|_0^2 \le C_2\, \frac{\delta t}{\varepsilon},$$

$$(v) \qquad \sum_{k=0}^{n} \left(\|\mathbf{v}^{k+1} - \widetilde{\mathbf{v}}^{k+1}\|_{-1}^2 + \|\mathbf{v}^{k+1} - \mathbf{v}^k\|_{-1}^2\right) \le 2C_0\, \frac{1+\lambda}{\lambda} = 2C_0\left(1 + \frac{\delta t}{\varepsilon}\right).$$

PROOF.
Proof of (i). With (47, 49, 51), we get the energy equations below since the solution of (47) is $\widetilde{\mathbf{v}}^{n+1} \in H^1_0(\Omega)^d$ with $\mathbf{v}_D = 0$, $\widehat{\mathbf{v}}^{n+1} \in H^1_n(\Omega)$ with Lemma 2, then $\mathbf{v}^{n+1} \in H^1_n(\Omega)$ and $b(\mathbf{v}^n, \widetilde{\mathbf{v}}^{n+1}, \widetilde{\mathbf{v}}^{n+1}) = 0$ from (46).

Moreover, with $p^0 \in H^1(\Omega)/\mathbb{R}$, we have $p^{n+1} \in H^1(\Omega)/\mathbb{R}$ for all $n \in \mathbb{N}$ such that $(n+1)\delta t \le T$, since $\nabla(p^{n+1} - p^n)$ belongs to $H^1_n(\Omega)$ with (49) or (13). Let us also notice that if $p^0 \in H^2(\Omega)/\mathbb{R}$, then we have $p^{n+1} \in H^2(\Omega)/\mathbb{R}$. This is due to the regularization effect of the penalty method, already encountered in Section 3. Then, using the Green formula by taking respectively the duality product of (47) with $2\delta t\, \widetilde{\mathbf{v}}^{n+1}$, the L^2 inner product of (49) with $2\delta t\, \mathbf{v}^{n+1}$, the L^2 inner product of (51) with $2\delta t\, p^{n+1}$ and the L^2 inner product of (49) with $2\delta t^2\,\nabla p^{n+1}$, we get for all $n\in\mathbb{N}$ such that $(n+1)\delta t\le T$:

$$\|\widetilde{\mathbf{v}}^{n+1}\|_0^2-\|\mathbf{v}^n\|_0^2+\|\widetilde{\mathbf{v}}^{n+1}-\mathbf{v}^n\|_0^2+\frac{2\delta t}{Re}\|\nabla\widetilde{\mathbf{v}}^{n+1}\|_0^2-2\delta t\left(p^n,\nabla\cdot\widetilde{\mathbf{v}}^{n+1}\right)_0 = 2\delta t\left\langle\mathbf{f}^{n+1},\widetilde{\mathbf{v}}^{n+1}\right\rangle_{-1}$$

$$\|\mathbf{v}^{n+1}\|_0^2-\|\widetilde{\mathbf{v}}^{n+1}\|_0^2+\|\mathbf{v}^{n+1}-\widetilde{\mathbf{v}}^{n+1}\|_0^2-2\delta t\left(p^{n+1}-p^n,\nabla\cdot\mathbf{v}^{n+1}\right)_0=0$$

$$\varepsilon\,\delta t\left(\|p^{n+1}\|_0^2-\|p^n\|_0^2+\|p^{n+1}-p^n\|_0^2\right)+2\delta t\left(p^{n+1},\nabla\cdot\mathbf{v}^{n+1}\right)_0=0$$

$$\delta t^2\left(\|\nabla p^{n+1}\|_0^2-\|\nabla p^n\|_0^2+\|\nabla(p^{n+1}-p^n)\|_0^2\right)-2\delta t\left(p^{n+1},\nabla\cdot(\mathbf{v}^{n+1}-\widetilde{\mathbf{v}}^{n+1})\right)_0=0.$$

Then, by summing the four previous equations, we get the following energy equality:

$$\begin{aligned}\|\mathbf{v}^{n+1}\|_0^2-\|\mathbf{v}^n\|_0^2\;+\;&\|\widetilde{\mathbf{v}}^{n+1}-\mathbf{v}^n\|_0^2+\|\mathbf{v}^{n+1}-\widetilde{\mathbf{v}}^{n+1}\|_0^2+\frac{2\delta t}{Re}\|\nabla\widetilde{\mathbf{v}}^{n+1}\|_0^2\\&+\;\delta t^2\left(\|\nabla p^{n+1}\|_0^2-\|\nabla p^n\|_0^2+\|\nabla(p^{n+1}-p^n)\|_0^2\right)\\&+\;\varepsilon\,\delta t\left(\|p^{n+1}\|_0^2-\|p^n\|_0^2+\|p^{n+1}-p^n\|_0^2\right)\\&-\;2\delta t\left(p^{n+1}-p^n,\nabla\cdot(\mathbf{v}^{n+1}-\widetilde{\mathbf{v}}^{n+1})\right)_0\\&=2\delta t\left\langle\mathbf{f}^{n+1},\widetilde{\mathbf{v}}^{n+1}\right\rangle_{-1}.\end{aligned}\tag{52}$$

Moreover, with the Green formula and using (49), we have the equality:

$$\begin{aligned}2\delta t\left(p^{n+1}-p^n,\nabla\cdot(\mathbf{v}^{n+1}-\widetilde{\mathbf{v}}^{n+1})\right)_0&=-2\delta t\left(\nabla(p^{n+1}-p^n),\mathbf{v}^{n+1}-\widetilde{\mathbf{v}}^{n+1}\right)_0\\&=2\,\|\mathbf{v}^{n+1}-\widetilde{\mathbf{v}}^{n+1}\|_0^2=2\delta t^2\,\|\nabla(p^{n+1}-p^n)\|_0^2\\&=\|\mathbf{v}^{n+1}-\widetilde{\mathbf{v}}^{n+1}\|_0^2+\delta t^2\,\|\nabla(p^{n+1}-p^n)\|_0^2.\end{aligned}\tag{53}$$

Let us now give some bound of the right-hand side term in (52). With the duality pairing, the Poincaré inequality and using $a\,b\le(a^2+b^2)/2$, we have:

$$\begin{aligned}2\delta t\left\langle\mathbf{f}^{n+1},\widetilde{\mathbf{v}}^{n+1}\right\rangle_{-1}&\le 2c(\Omega)\,\delta t\,\|\mathbf{f}^{n+1}\|_{-1}\,\|\nabla\widetilde{\mathbf{v}}^{n+1}\|_0\\&\le\frac{\delta t}{Re}\|\nabla\widetilde{\mathbf{v}}^{n+1}\|_0^2+c(\Omega)^2\,Re\,\delta t\,\|\mathbf{f}^{n+1}\|_{-1}^2.\end{aligned}$$

By using this bound in (52) and summing with (53), we get:

$$\begin{aligned}\|\mathbf{v}^{n+1}\|_0^2-\|\mathbf{v}^n\|_0^2\;+\;&\|\widetilde{\mathbf{v}}^{n+1}-\mathbf{v}^n\|_0^2+\frac{\delta t}{Re}\|\nabla\widetilde{\mathbf{v}}^{n+1}\|_0^2\\&+\;\delta t^2\left(\|\nabla p^{n+1}\|_0^2-\|\nabla p^n\|_0^2\right)\\&+\;\varepsilon\,\delta t\left(\|p^{n+1}\|_0^2-\|p^n\|_0^2+\|p^{n+1}-p^n\|_0^2\right)\\&\le c(\Omega)^2\,Re\,\delta t\,\|\mathbf{f}^{n+1}\|_{-1}^2.\end{aligned}$$

We now write the previous inequality with the index k instead of n, and then sum it for $k=0,\dots,n$. This yields the following energy estimate for all $n\in\mathbb{N}$ such that $(n+1)\delta t\le T$:

$$\begin{aligned}\|\mathbf{v}^{n+1}\|_0^2\;+\;&\sum_{k=0}^{n}\|\widetilde{\mathbf{v}}^{k+1}-\mathbf{v}^k\|_0^2+\frac{1}{Re}\sum_{k=0}^{n}\delta t\,\|\nabla\widetilde{\mathbf{v}}^{k+1}\|_0^2\\&+\;\delta t^2\,\|\nabla p^{n+1}\|_0^2+\varepsilon\,\delta t\,\|p^{n+1}\|_0^2+\varepsilon\sum_{k=0}^{n}\delta t\,\|p^{k+1}-p^k\|_0^2\\&\le\|\mathbf{v}^0\|_0^2+\varepsilon\,\delta t\,\|p^0\|_0^2+\delta t^2\,\|\nabla p^0\|_0^2+c(\Omega)^2\,Re\sum_{k=0}^{n}\delta t\,\|\mathbf{f}^{k+1}\|_{-1}^2\\&\le C_0\end{aligned}\tag{54}$$

which concludes the proof of (i) with $\varepsilon \leq 1$ and $\delta t \leq T$.

Proof of (ii). From (14) or (51), we have: $\nabla \cdot \mathbf{v}^{n+1} = -\varepsilon\,(p^{n+1} - p^n)$. Hence, using the bound in (54) or (i), this immediately yields (ii) with:

$$\sum_{k=0}^{n} \delta t\,\|\nabla \cdot \mathbf{v}^{k+1}\|_0^2 = \sum_{k=0}^{n} \delta t\,\|p^{k+1} - p^k\|_0^2\,\varepsilon^2 \leq C_0\,\varepsilon. \tag{55}$$

Proof of (iii). The previous estimate (i) immediately gives: $\|\nabla \pi^{n+1}\|_0^2 \leq C_0$ by denoting $\pi^{n+1} = \delta t\,p^{n+1}$. Then, the bound (iii) is obtained by using the mean Poincaré inequality: $\|\pi^{n+1}\|_0 \leq c_0(\Omega)\,\|\nabla \pi^{n+1}\|_0$ since π^{n+1} belongs to $L_0^2(\Omega) \cap H^1(\Omega)$.

Proof of (iv_a) for a simply connected domain Ω. We recall from Lemma 2 that $\widehat{\mathbf{v}}^{n+1} = (\mathbf{v}^{n+1} - \widetilde{\mathbf{v}}^{n+1}) \in H_n^1(\Omega)$ with zero curl since the space $H_{div,0}(\Omega) \cap H_{rot}(\Omega)$ is continuously imbedded in $H_n^1(\Omega)$, see [40,3]. Then, with the H^1-norm $\|.\|_1$ equivalence to the norm $\left(\|div.\|_0^2 + \|curl.\|_0^2\right)^{\frac{1}{2}}$ for functions in $H_n^1(\Omega)$ when the domain Ω is simply connected, see [35,38], there exists $c_1(\Omega) > 0$ such that:

$$\|\mathbf{v}^{n+1} - \widetilde{\mathbf{v}}^{n+1}\|_1 = \|\widehat{\mathbf{v}}^{n+1}\|_1 \leq c_1(\Omega)\,\|\nabla \cdot \widehat{\mathbf{v}}^{n+1}\|_0 \leq c_1(\Omega)\left(\|\nabla \cdot \mathbf{v}^{n+1}\|_0 + \|\nabla \cdot \widetilde{\mathbf{v}}^{n+1}\|_0\right).$$

Then, using the fact that: $\|\nabla \cdot \widetilde{\mathbf{v}}^{n+1}\|_0 \leq \|\nabla \widetilde{\mathbf{v}}^{n+1}\|_0$ since $\widetilde{\mathbf{v}}^{n+1} \in H_0^1(\Omega)^d$ and the inequality $(a+b)^2 \leq 2\,(a^2+b^2)$, we get with the previous bounds in (54,55):

$$\begin{aligned}\sum_{k=0}^{n} \delta t\,\|\mathbf{v}^{k+1} - \widetilde{\mathbf{v}}^{k+1}\|_1^2 &\leq 2c_1(\Omega)^2 \left(\sum_{k=0}^{n} \delta t\,\|\nabla \cdot \mathbf{v}^{k+1}\|_0^2 + \sum_{k=0}^{n} \delta t\,\|\nabla \widetilde{\mathbf{v}}^{k+1}\|_0^2\right) \\ &\leq 2c_1(\Omega)^2\,C_0\,(\varepsilon + Re)\,.\end{aligned} \tag{56}$$

Finally, using the triangular inequality: $\|\nabla \mathbf{v}^{k+1}\|_0 \leq \|\nabla(\mathbf{v}^{k+1} - \widetilde{\mathbf{v}}^{k+1})\|_0 + \|\nabla \widetilde{\mathbf{v}}^{k+1}\|_0$, and the bounds in (54,56), yields (iv_a) with:

$$\begin{aligned}&\sum_{k=0}^{n} \delta t\,\|\nabla \mathbf{v}^{k+1}\|_0^2 + \sum_{k=0}^{n} \delta t\,\|\nabla(\mathbf{v}^{k+1} - \widetilde{\mathbf{v}}^{k+1})\|_0^2 \\ &\qquad \leq C_1\left(\Omega, T, Re, \|\mathbf{f}\|_{L^2(0,T;H^{-1})}, \|\mathbf{v}^0\|_0, \|p^0\|_1\right).\end{aligned} \tag{57}$$

Proof of (iv_b). When the domain Ω is not simply connected, the H^1-norm $\|.\|_1$ is equivalent to the norm $\left(\|.\|_0^2 + \|div.\|_0^2 + \|curl.\|_0^2\right)^{1/2}$. By a simple energy estimate with (48), we have :

$$\|\widehat{\mathbf{v}}^{n+1}\|_0^2 + \frac{\delta t}{2\varepsilon}\,\|\nabla \cdot \widehat{\mathbf{v}}^{n+1}\|_0^2 \leq \frac{\delta t}{2\varepsilon}\,\|\nabla \cdot \widetilde{\mathbf{v}}^{n+1}\|_0^2.$$

Thus we have with $\|\nabla \cdot \widetilde{\mathbf{v}}^{n+1}\|_0 \leq \|\nabla \widetilde{\mathbf{v}}^{n+1}\|_0$:

$$\|\widehat{\mathbf{v}}^{n+1}\|_1^2 := \|\mathbf{v}^{n+1} - \widetilde{\mathbf{v}}^{n+1}\|_1^2 \leq c_1(\Omega)\left(1 + \frac{\delta t}{2\varepsilon}\right)\|\nabla \widetilde{\mathbf{v}}^{n+1}\|_0^2.$$

Hence, we get (iv_b) using the bound (i):

$$\sum_{k=0}^{n} \delta t\,\|\nabla \mathbf{v}^{k+1}\|_0^2 + \sum_{k=0}^{n} \delta t\,\|\nabla(\mathbf{v}^{k+1} - \widetilde{\mathbf{v}}^{k+1})\|_0^2 \leq C_2\,\frac{\delta t}{\varepsilon}.$$

Proof of (v). With (48), we have:

$$\widehat{\mathbf{v}}^{n+1} = \mathbf{v}^{n+1} - \widetilde{\mathbf{v}}^{n+1} = \frac{\delta t}{\varepsilon} \nabla \left(\nabla \cdot \mathbf{v}^{n+1} \right)$$

and using the definition of the H^{-1} norm, we get:

$$\| \mathbf{v}^{n+1} - \widetilde{\mathbf{v}}^{n+1} \|_{-1} = \frac{\delta t}{\varepsilon} \| \nabla \left(\nabla \cdot \mathbf{v}^{n+1} \right) \|_{-1} \leq \frac{\delta t}{\varepsilon} \| \nabla \cdot \mathbf{v}^{n+1} \|_{0}.$$

Hence, using the divergence bound in (ii) and assuming that $\varepsilon = \lambda\, \delta t$ for a given constant $\lambda > 0$, it yields:

$$\begin{aligned} \sum_{k=0}^{n} \| \mathbf{v}^{k+1} - \widetilde{\mathbf{v}}^{k+1} \|_{-1}^{2} &\leq \frac{\delta t}{\varepsilon^{2}} \sum_{k=0}^{n} \delta t \, \| \nabla \cdot \mathbf{v}^{k+1} \|_{0}^{2} \\ &\leq C_0 \frac{\delta t}{\varepsilon} = \frac{C_0}{\lambda}. \end{aligned} \tag{58}$$

Then, writing $\mathbf{v}^{k+1} - \mathbf{v}^{k} = (\mathbf{v}^{k+1} - \widetilde{\mathbf{v}}^{k+1}) + (\widetilde{\mathbf{v}}^{k+1} - \mathbf{v}^{k})$ and using the triangular inequality, we get the bound (v) with the canonical injection of $L^2(\Omega)^d$ inside $H^{-1}(\Omega)^d$ and the bound in (i):

$$\begin{aligned} \sum_{k=0}^{n} \| \mathbf{v}^{k+1} - \mathbf{v}^{k} \|_{-1}^{2} &\leq \sum_{k=0}^{n} \left(\| \mathbf{v}^{k+1} - \widetilde{\mathbf{v}}^{k+1} \|_{-1} + \| \widetilde{\mathbf{v}}^{k+1} - \mathbf{v}^{k} \|_{-1} \right)^{2} \\ &\leq 2 \sum_{k=0}^{n} \left(\| \mathbf{v}^{k+1} - \widetilde{\mathbf{v}}^{k+1} \|_{-1}^{2} + \| \widetilde{\mathbf{v}}^{k+1} - \mathbf{v}^{k} \|_{-1}^{2} \right) \\ &\leq 2\, C_0 \frac{1+\lambda}{\lambda} = 2\, C_0 \left(1 + \frac{\delta t}{\varepsilon} \right). \end{aligned} \tag{59}$$

This completes the proof of the theorem. □

Remark 6 (Unconditional stability of the fully implicit (VPP$_\varepsilon$) method.)

If we consider the non-linear implicit (VPP$_\varepsilon$) method by replacing the linearized implicit convection term $B\left(\mathbf{v}^{n}, \widetilde{\mathbf{v}}^{n+1}\right)$ in the prediction step (47) either by $B\left(\widetilde{\mathbf{v}}^{n+1}, \widetilde{\mathbf{v}}^{n+1}\right)$ or $B\left(\mathbf{v}^{n+1}, \widetilde{\mathbf{v}}^{n+1}\right)$, the previous Theorem 3 of unconditional stability still holds. Indeed, with (46) we have both $b(\widetilde{\mathbf{v}}^{n+1}, \widetilde{\mathbf{v}}^{n+1}, \widetilde{\mathbf{v}}^{n+1}) = 0$ and $b(\mathbf{v}^{n+1}, \widetilde{\mathbf{v}}^{n+1}, \widetilde{\mathbf{v}}^{n+1}) = 0$, and hence the stability proof is exactly the same as in Theorem 3.

Remark 7 (Convergence to Navier-Stokes solutions.)

By using compactness arguments from Aubin-Lions-Simon [55,70] or with Kolmogorov's theorem [21,20], since we have a bound of the dissipation with the time translates in (v), Theorem 3 allows us to prove the convergence of the (VPP$_\varepsilon$) solution to Navier-Stokes solutions when $\varepsilon = \delta t$ tends to zero, without additional regularity assumption. Moreover, the convergence can be also carried out when $\delta t \to 0$ for a fixed value of $\varepsilon > 0$. These convergence analyses will deserve a further study.

Remark 8 (Splitting error of the (VPP$_\varepsilon$) method.)

We prove in Theorem 3 (ii) that the L^2-norm of the velocity divergence at least vanishes as $\mathscr{O}(\sqrt{\varepsilon})$. Besides, the previous works on first versions of the VPP method [7,9] suggest that the $l^2(0,T;L^2(\Omega))$ norm of the velocity divergence vanishes as $\mathscr{O}(\varepsilon\, \delta t)$, *i.e.* as $\mathscr{O}(\gamma M^2 / \rho\, V^{\star 2})$ with (26). This will be rigourously proved in a further paper, but it is already confirmed by the numerical results in [10]. It also suggests that the $l^2(0,T;L^2(\Omega)^d)$ norm of the velocity splitting error vanishes as $\mathcal{O}(\varepsilon\,\delta t)$, and that the $l^\infty(0,T;L^2(\Omega)^d)$ and $l^2(0,T;H^1(\Omega)^d)$ norms of the velocity splitting error vanish as $\mathcal{O}(\varepsilon\,\delta t^{\frac{3}{4}})$ whereas the norm in $l^2(0,T;L^2(\Omega))$ of the pressure splitting error vanishes as $\mathcal{O}(\varepsilon\,\delta t^{\frac{1}{2}})$; see [9, Theorem 2.4] for more details. This is also in agreement with the estimates proved in [18, Theorem 4.5] for the scalar penalty-projection method [47,48,37] where the augmentation parameter r, playing here the role of $1/\varepsilon$, takes large values. By choosing only $\varepsilon = \delta t$, these splitting errors are thus already of the same order of those known for the best incremental projection methods. Up to our knowledge, the splitting errors of the incremental projection method in the case of the Stokes problem with Dirichlet boundary conditions scale as $\mathcal{O}(\delta t^{\frac{3}{2}})$ in the $l^\infty(0,T;L^2(\Omega)^d)$ norm and $\mathcal{O}(\delta t^2)$ in $l^2(0,T;L^2(\Omega)^d)$ norm for the velocity and $\mathcal{O}(\delta t^{\frac{3}{2}})$ for the pressure in $l^2(0,T;L^2(\Omega))$ norm; see [44] and the references therein for the details.

However, the splitting errors of the (VPP$_\varepsilon$) method can be made as small as desired until the machine precision, and thus completely negligible with respect to the time consistency error of the scheme, *i.e.* $\mathcal{O}(\delta t)$ in the present case or $\mathcal{O}(\delta t^2)$ for second-order (VPP$_\varepsilon$) methods [14,15], by choosing $\varepsilon = \delta t^2$ or $\varepsilon = \delta t^3$. This is also another nice advantage of our method.

5 Conclusion and perspectives

We have described in detail and analysed the fast vector penalty-projection method (VPP$_\varepsilon$) for the solution of both homogeneous or non-homogeneous or also multiphase incompressible Navier-Stokes/Brinkman problems.

The method is also numerically validated in [10] where several benchmark problems concerning homogeneous, dilatable, non-homogeneous or multiphase flows are investigated and where we compare it to the Uzawa augmented Lagrangian and scalar incremental projection methods. More recently, a severe multiphase benchmark with strong stresses has been computed where the mass density ratio is equal to 8000 with a large capillarity constant and corresponding to air bubble dynamics in a melted liquid steel [13]. Nevertheless for dispersed bubbly flows, it is difficult to compare our numerical method with others since most of them have difficulties to compute results with a suitable mesh convergence when the mass density ratio exceeds several hundreds. However, to evaluate and validate the robustness of the (VPP$_\varepsilon$) method with respect to large density or viscosity ratios, we have computed the motion of an heavy solid ball of constant density $\rho_s = 10^6$ which freely falls vertically in air with the gravity force $\mathbf{f} = \rho_s\,\mathbf{g}$, and corresponding to a low-Mach number flow with $\varepsilon = 10^{-6}$ and $\delta t = 2\,10^{-4}$. The rigid motion of the body is obtained by letting the viscosity μ_s tend to infinity inside the ball in order to penalize the tensor of deformation rate $\mathbf{d}(\mathbf{v})$. We refer to [8,10] for the numerical results of this test case where the mass density ratio equals 10^6 and the viscosity ratio equals 10^{17}. Hence, this benchmark for fluid-structure interaction problems allows us to assess the robustness and versatility of the method for large density or viscosity ratios. Indeed, a very nice and important feature of the (VPP$_\varepsilon$) method is that, contrary to all other splitting methods, its robustness is not sensitive to large mass density ratios since the velocity penalty-projection step does not include any spatial derivative of the density. The method is also validated to compute anisotropic and heterogeneous Darcy problems with large permeability ratios in [12].

The (VPP$_\varepsilon$) method finally proves to be really promising since it is fast, cheap, and robust whatever the density, viscosity or anisotropic permeability jumps. Indeed, our method can efficiently and accurately compute some severe test cases, whereas other famous methods either fail or cannot reach the suitable mesh convergence and run slower.

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